INTRODUCTION

Purpose of Document

This document provides specific learning outcomes and achievement indicators for planning of scope, sequence, and depth for both Introduction to Calculus (45S) and Advanced Mathematics (45S or 40S). This document is intended to communicate high expectations for students' mathematical learning to all education partners.

A student can earn up to three different optional half-credit mathematics courses (or equivalent full-credit courses). These optional half-credit courses are designed to meet the needs of students who demonstrate particular aptitude or strong interest in mathematics and who desire to study more advanced topics. These courses are intended to assist students in their transition from secondary to post-secondary mathematics courses.

These optional half-credit courses provide an introduction to areas of mathematics that are covered in post-secondary programs within Manitoba and out of province. These courses will be most helpful for those students planning to study engineering, mathematics, science, computer science, or other mathematics-oriented programs. When choosing the optional half courses and topics within them, students should consider their interests, both future and current. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study, as they vary by institution and by year.

Beliefs about Students and Mathematics Learning

Students are curious, active learners with individual interests, abilities, needs, and career goals. They come to school with varying knowledge, life experiences, expectations, and backgrounds. A key component in developing students' mathematical literacy is making connections to these backgrounds, experiences, goals, and aspirations. Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences.

This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex, and that connect concrete and abstract representations. The use of manipulatives, visuals, and a variety of pedagogical and assessment approaches can address the diversity of learning styles. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics.

Students need frequent opportunities to develop and reinforce their conceptual understanding, procedural thinking, and problem-solving abilities. By addressing these three interrelated components, students will strengthen their ability to apply mathematical learning to their daily lives.

The learning environment should value, respect, and address all students' experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary, depending upon how the problem is understood.

Assessment *for* learning, assessment *as* learning, and assessment *of* learning are all critical to helping students learn mathematics. A variety of evidence and a variety of assessment approaches should be used in the mathematics classroom.

First Nations, Métis, and Inuit Perspectives

First Nations, Métis, and Inuit students in Manitoba come from diverse geographic areas and have varied cultural and linguistic backgrounds. Students attend schools in a variety of settings, including urban, rural, and isolated communities. Teachers need to recognize and understand the diversity of cultures within schools and the diverse experiences of students.

First Nations, Métis, and Inuit students often have a whole-world view of the environment; as a result, many of these students live and learn best in a holistic way. This means that students look for connections in learning and learn mathematics best when it is contextualized and not taught as discrete content.

Many First Nations, Métis, and Inuit students come from cultural environments where learning takes place through active hands-on participation. Traditionally, little or no emphasis was placed upon the written word. Oral communication, along with practical applications and experiences, is important to student learning and understanding.

A variety of teaching and assessment strategies is required to build upon the diverse knowledge, cultures, communication styles, skills, attitudes, experiences, and learning styles of students.

The strategies used must go beyond the incidental inclusion of topics and objects unique to a culture or region and strive to achieve higher levels of multicultural education (Banks and Banks).

Affective Domain

A positive attitude is an important aspect of the affective domain, which has a profound effect on learning. Environments that create a sense of belonging, support risk taking, and provide opportunities for success help students to develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, to participate willingly in classroom activities, to persist in challenging situations, and to engage in reflective practices.

Teachers, students, and parents need to recognize the relationship between the affective and cognitive domains and to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Goals for Students

The main goals of mathematics education are to prepare students to

- communicate and reason mathematically
- use mathematics confidently, accurately, and efficiently to solve problems
- appreciate and value mathematics
- make connections between mathematical knowledge and skills and their applications
- commit themselves to lifelong learning
- become mathematically literate citizens, using mathematics to contribute to society and to think critically about the world

Students who have met these goals

- gain an understanding and appreciation of the role of mathematics in society
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical problem solving
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history

Mathematical Processes

The seven mathematical processes are critical aspects of learning, doing, and understanding mathematics. Students must encounter these processes regularly in a mathematics program in order to achieve the goals of mathematics education. The following interrelated mathematical processes are intended to permeate the teaching and learning of mathematics. Students are expected to

- use communication in order to learn and express their understanding
- make connections among mathematical ideas, other concepts in mathematics, everyday experiences, and other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technology as a tool for learning and solving problems
- develop visualization skills to assist in processing information, making connections, and solving problems

All seven processes should be used in the teaching and learning of mathematics.

Communication

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links among their own language and ideas, the language and ideas of others, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology. Communication can play a significant role in helping students make connections among concrete, pictorial, graphical, symbolic, verbal, written, and mental representations of mathematical ideas. Explanations of ideas should include the various representations as appropriate. Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Connections

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections. “Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding.... Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching” (Caine and Caine 5).

Mental Mathematics and Estimation

Mental mathematics and estimation is a combination of cognitive strategies that enhance flexible thinking and number sense. It involves using strategies to perform mental calculations.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy, and flexibility in reasoning and calculating. “Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math” (NCTM, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein 442).

Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers” (Hope v).

Estimation is used for determining approximate values or quantities, usually by referring to benchmarks or referents, or for determining the reasonableness of calculated values. Estimation is also used to make mathematical judgments and to develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it. Developing number sense is a lifelong process. Students benefit from opportunities to practise and solidify previously learned skills and procedures in new contexts.

Problem Solving

”Problem solving is an integral part of all mathematics learning” (NCTM, 2000). Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type “How would you...?” or “How could you...?”, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be based on problem solving, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately.

A true problem requires students to use prior knowledge in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks.

Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple creative and innovative solutions. Creating an environment where students openly look for and engage in the process of finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Reasoning

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them to develop an understanding of mathematics. All students need to be challenged to answer questions such as "Why do you believe that's true/correct?" or "What would happen if...?".

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

When explaining ideas, students should be encouraged to use concrete, pictorial, symbolic, graphical, verbal, and written representations of their mathematical ideas.

Technology

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical learning outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures, and solve problems. Some post-secondary institutions expect fluency with technology.

Technology has the potential to enhance the teaching and learning of mathematics. It can be used to

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense
- create geometric figures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. Students need to know when it is appropriate to use technology such as a calculator and when to apply their mental computation, reasoning, and estimation skills to predict and check answers. The use of technology can enhance, although it should not replace, conceptual understanding, procedural thinking, and problem solving.

Visualization

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret, and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate, and it involves knowledge of several estimation strategies (Shaw and Cliatt 150).

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Nature of Mathematics

Mathematics is one way of understanding, interpreting, and describing our world. There are a number of characteristics that define the nature of mathematics, including change, constancy, number sense, patterns, relationships, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12... can be described as

- skip-counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain (Steen 184)

Constancy

Many important properties in mathematics do not change when conditions change. Examples of constancy include

- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- theoretical probability of an event

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems such as those involving constant rates of change, lines with constant slope, or direct variation situations.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

Number sense is an awareness and understanding of what numbers are, their relationships, their magnitude, and the relative effect of operating on numbers, including the use of mental mathematics and estimation (Fennell and Landis 187).

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics. Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment.

Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns, and describing possible relationships visually, symbolically, orally, or in written form. Technology should be used to aid in the search for relationships.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. Some of these experiences should involve the use of technology. These experiences offer a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations. Graphing calculators or graphing software can aid students in developing this understanding.

Uncertainty

In mathematics, interpretations of data and the predictions made from data inherently lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important that students recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of an interpretation or conclusion is directly related to the quality of the data it is based upon. An awareness of uncertainty provides students with an understanding of why and how to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Use of Mathematics

It is important to study mathematics because of its many applications to everyday life and to specialized areas of work. A knowledge of mathematics is necessary if students are to have an awareness and an understanding of their environment and to enjoy a fuller life. With this in mind, it is suggested that mathematics be considered in the following ways:

- **As a Mode of Communication**

  Mathematics is a language. For students to communicate successfully in mathematics, they must learn the basic concepts without having to meet demands for logical rigour. At first, the language acquired should be used to describe patterns and relationships observed in the real world, and it should allow students to share their mathematical thinking. At a later stage, symbols should be introduced to record these relationships, expand mathematical vocabulary, and communicate understanding.

- **As a Tool**

  Mathematics is a tool used to answer practical questions, thus demonstrating to students that to function in society they need a good understanding of arithmetic, algebra, and statistics. Having a deeper understanding of mathematics allows students to solve problems in different ways, see mathematics concepts from a variety of perspectives, develop a broad range of mathematical tools, and apply the tools appropriately and efficiently. Students should become capable of employing an organized and sometimes creative process leading to the solution of a problem. This may involve applying fundamental operations and information-processing skills—such as collecting, organizing, and interpreting data—to familiar and unfamiliar contexts. Realistic problems relating to the students' environment can be engaging for some students as they make connections to contexts they may encounter in the future.

- **As a Source of Aesthetic Enjoyment**

  Mathematics is an aesthetic discipline like music. Many students can develop an appreciation of mathematics through the discovery of patterns in numbers and in created or naturally occurring designs. An elegant solution to a mathematics problem can be aesthetically pleasing. Methodologies and techniques conducive to developing positive inquiring attitudes toward mathematics should be employed.

Students should be given opportunities to recognize and appreciate these three ways in which the subject can be used. Classroom emphasis on these uses cannot always be equal, however, and it must vary depending on the nature of the topic and the interest and abilities of the students.