Teaching Mental Math and Estimation Strategies

Teaching Mental Math and Estimation Strategies

Dozens of mental math and estimation strategies exist, and many go by several different names. This document provides a limited number of strategies and in no way should be taken as an exhaustive list of all of the mental math and estimation strategies available. Strategies can be found online or through teacher guides, and can even be developed within your own classes.

Curriculum and This Support Document

The Manitoba curriculum does not set specific grades as required teaching for many of these strategies. Some Manitoba divisions have found success in dividing up the most common strategies for specific grade-level focus. Although most of the strategies can be used at all grade levels, organizing a cross-grade method of focusing on common strategies may reduce the amount of focus that teachers feel they need to place on teaching each of the many strategies.

BLM 5-8.8, from the Grades 5–8 Blackline Masters package, is attached in order to help you and your school develop an approach to focusing on several of these strategies while aligning with specific grade-level curricular outcomes.

Strategy posters and teaching methods are also attached for teacher and student support.

This Grade 8 Mental Math document is designed as a support to the existing curriculum and is in no way a mandated program. Individual teachers should use it as they see fit.

Topics

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BLM 5–8.8: Mental Math Strategies

The following list compiles mental math strategies as found in *Kindergarten to Grade 8 Mathematics: Manitoba Curriculum Framework of Outcomes*. Note: This resource is meant for teacher information, not as a list of strategies that students should memorize.

Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7
1.N.10.	2.N.8. 2.N.10.	3.N.6. 3.N.7. 3.N.10. 3.N.11. 3.N.12.	4.N.4. 4.N.5. 4.N.6. 4.N.11.	5.N.2. 5.N.3. 5.N.4.	6.N.8.	7.N.2.

Grade	Concept	Strategy	Meaning	Example
1	Addition	Counting on	Students begin with a number and count on to get the sum. Students should begin to recognize that beginning with the larger of the two addends is generally most efficient.	*for* 3 + 5 *think* 5 + 1 + 1 + 1 *is* 8; *think* 5, 6, 7, 8
1	Subtraction	Counting back	Students begin with the minuend and count back to find the difference.	*for* 6 − 2 *think* 6 − 1 − 1 *is* 4; *think* 6, 5, 4
1, 2	Addition	Using one more	Starting from a known fact and adding one more.	*for* 8 + 5 *if you know* 8 + 4 *is* 12 *and one more is* 13
1, 2	Addition	Using one less	Starting from a known fact and taking one away.	*for* 8 + 6 *if you know* 8 + 7 *is* 15 *and one less is* 14
1, 2,	Addition Subtraction	Making 10	Students use combinations that add up to ten and can extend this to multiples of ten in later grades.	4 + ____ *is* 10 7 + ____ *is* 10; *so* 23 + ____ *is* 30

(continued)

BLM 5–8.8: Mental Math Strategies (Continued)

Grade	Concept	Strategy	Meaning	Example
1	Addition Subtraction	Starting from known doubles	Students need to work to know their doubles facts.	2 + 2 *is* 4 *and* 4 − 2 *is* 2
1, 2, 3	Subtraction	Using addition to subtract	This is a form of part-part-whole representation. Thinking of addition as: part + part = whole Thinking of subtraction as: whole − part = part	*for* 12 − 5 *think* 5 + ____ = 12 *so* 12 − 5 *is* 7
2	Addition Subtraction	The zero property of addition	Knowing that adding 0 to an addend does not change its value, and taking 0 from a minuend does not change the value.	0 + 5 = 5; 11 − 0 = 11
2, 3	Addition Subtraction	Using doubles	Students learn doubles, and use this to extend facts: using doubles doubles plus one (or two) doubles minus one (or two)	*for* 5 + 7 *think* 6 + 6 *is* 12; *for* 5 + 7 *think* 5 + 5 + 2 *is* 12 *for* 5 + 7 *think* 7 + 7 − 2 *is* 12
2, 3	Addition Subtraction	Building on known doubles	Students learn doubles, and use this to extend facts.	*for* 7 + 8 *think* 7 + 7 *is* 14 *so* 7 + 8 *is* 14 + 1 *is* 15
3	Addition	Adding from left to right	Using place value understanding to add 2-digit numerals.	*for* 25 + 33 *think* 20 + 30 *and* 5 + 3 *is* 50 + 8 *or* 58

(continued)

BLM 5–8.8: Mental Math Strategies (Continued)

Grade	Concept	Strategy	Meaning	Example
3	Addition Subtraction	Making 10	Students use combinations that add up to ten to calculate other math facts and can extend this to multiples of ten in later grades.	*for* 8 + 5 *think* 8 + 2 + 3 *is* 10 + 3 *or* 13
3	Addition Subtraction	Compensation	Using other known math facts and compensating. For example, adding 2 to an addend and taking 2 away from the sum.	*for* 25 + 33 *think* 25 + 35 − 2 *is* 60 − 2 *or* 58
3	Addition	Commutative property	Switching the order of the two numbers being added will not affect the sum.	4 + 3 *is the same as* 3 + 4
3, 4 (decimals)	Addition Subtraction	Compatible numbers	Compatible numbers are friendly numbers (often associated with compatible numbers to 5 or 10).	*for* 4 + 3 *students may think* 4 + 1 *is* 5 *and* 2 *more makes* 7
3	Multiplication Division	Array	Using an ordered arrangement to show multiplication or division (similar to area).	*for* 3 × 4 *think* • • • • • • • • • • • • *for* 12 ÷ 3 *think* • • • • • • • • • • • •
3	Multiplication	Commutative property	Switching the order of the two numbers being multiplied will not affect the product.	4 × 5 *is the same as* 5 × 4
3	Multiplication	Skip-counting	Using the concept of multiplication as a series of equal grouping to determine a product.	*for* 4 × 2 *think* 2, 4, 6, 8 *so* 4 × 2 *is* 8
4	Multiplication	Zero property of multiplication	Multiplying a factor by zero will always result in zero.	30 × 0 *is* 0 0 × 15 *is* 0

(continued)

BLM 5–8.8: Mental Math Strategies (Continued)

Grade	Concept	Strategy	Meaning	Example
4	Multiplication Division	Multiplicative identity	Multiplying (dividing) a factor (dividend) by one will not change its value.	1 × 12 *is* 12 21 ÷ 1 *is* 21
4, 5	Multiplication Division	Skip-counting from a known fact	Similar to the counting on strategy for addition. Using a known fact and skip-counting forward or backward to determine the answer.	*for* 3 × 8 *think* 3 × 5 is 15 *and skip-count by threes* 15, 18, 21, 24
4, 5	Multiplication Division	Doubling or halving	Using known facts and doubling or halving them to determine the answer.	*for* 7 × 4, *think the double of* 7 × 2 *is* 28 *for* 48 ÷ 6, *think the double of* 24 ÷ 6 *is* 8
4	Multiplication Division	Using the pattern for 9s	Knowing the first digit of the answer is one less than the non-nine factor and the sum of the product's digits is nine.	*for* 7 × 9 *think one less than* 7 *is* 6 *and* 6 *plus* 3 *is nine, so* 7 × 9 *is* 63
4, 5	Multiplication	Repeated doubling	Continually doubling to get to an answer.	*for* 3 × 8, *think* 3 × 2 *is* 6, 6 × 2 *is* 12, 12 × 2 *is* 24
4	Division	Using multiplication to divide	This is a form of part-part-whole representation. Thinking of multiplication as: part × part = whole Thinking of division as: whole ÷ part = part	*for* 35 ÷ 7 *think* 7 × ____ = 35 *so* 35 ÷ 7 *is* 5
4, 5	Multiplication	Distributive property	In arithmetic or algebra, when you distribute a factor across the brackets: $a \times (b + c) = a \times b + a \times c$ $(a + b) \times (c + d) = ac + ad + bc + bd$	*for* 2 × 154 *think* 2 × 100 *plus* 2 × 50 *plus* 2 × 4 *is* 200 + 100 + 8 *or* 308

(continued)

BLM 5–8.8: Mental Math Strategies (Continued)

Grade	Concept	Strategy	Meaning	Example
5	Division	Repeated halving	Continually halving to get a number.	*for* 32 ÷ 4, *think* 32 ÷ 2 is 16 *and* 16 ÷ 2 *is* 8 *so* 32 ÷ 4 *is* 8
5	Multiplication	Annexing zeros	When multiplying by a factor of 10 (or a power of ten), taking off the zeros to determine the product and adding them back on.	*for* 4 × 700, *think* 4 × 7 *is* 28 *and add two zeros to make* 2800
5	Multiplication	Halving and doubling	Halving one factor and doubling the other.	*for* 24 × 4, *think* 48 × 2 *is* 96
6, 7	Division	Dividing by multiples of ten	When dividing by 10, 100, etc., the dividend becomes smaller by 1, 2, etc. place-value positions.	*for* 76.3 ÷ 10 *think* 76.3 *should become smaller by one place-value position so* 76.3 ÷ 10 *is* 7.63

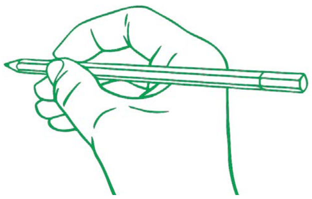
Mental Math

Grade 8 Mathematics

S–1

Sample Strategy

Standard algorithms and mental math



Although students need to become proficient in performing standard algorithms on paper, their cumbersome nature make them difficult for use with mental math.

EXAMPLE 1

$$\begin{array}{r} {}^{1\,1} \\ 173 \\ +\ \ 67 \\ \hline 240 \end{array}$$

7 + 3 = 10, remember 0, carry 1
1 + 7 + 6 = 14, remember 4, carry 1
1 + 1 = 2, remember 2

Reverse 0, 4, 2 to 2, 4, 0 and mentally reassign place value to **240**.

EXAMPLE 2

$$\begin{array}{r} \$91 \\ \times\ \ 13 \\ \hline 273 \\ 910 \\ \hline \$1183 \end{array}$$

1 × 3 = 3, remember 3
9 × 3 = 27, remember 27

Rearrange mentally into 273.

Add a place-holder zero.

1 × 1 = 1, remember 1
9 × 1 = 9, remember 9

Rearrange mentally into 910.

3 + 0 = 3, remember 3
7 + 1 = 8, remember 8
2 + 9 = 11, remember 11

Rearrange mentally and reassign place value and units to **1183**.

Mental mathematics strategies need to be analyzed for efficiency. In most cases, standard algorithms are very useful for paper-and-pencil arithmetic but are not efficient for doing mental calculations.

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–1

S–1

Standard algorithms and mental math

The standard algorithms for addition, subtraction, multiplication, and division work efficiently for paper-and-pencil tasks and should be encouraged for use with these. For mental math, however, using the standard algorithms often requires more steps than other methods outlined in this document.

The goals of mental math and estimation, as outlined by the Manitoba curriculum, are to enhance flexible thinking and number sense, and to improve computational fluency by developing efficiency, accuracy, and flexibility. It is important that students have a variety of methods at their disposal in order to choose the method that will, for them, be the most efficient and accurate method. It is also important that they have a conceptual understanding of the method they use in order to gain flexibility. For these reasons, it is necessary that students have additional tools for solving math mentally.

Students, and even some adults, find standard algorithms to be the most effective mental math strategy because of their own familiarity with them. Although these methods may not have the flexibility or efficiency of other methods, and the large number of steps increases the likelihood of error, standard algorithms may be an effective, although inefficient, approach for some in the same way that skip-counting for multiplication or counting back for subtraction can be effective, but inefficient.

Students should be encouraged to find their own most efficient and least cumbersome method of performing mental math in a variety of situations. They can learn this through both direct and indirect teaching methods.

> "Although results differ depending on what and how manipulatives are used in learning situations, learning with manipulatives is correlated positively with later development of mental mathematics (Gravemeijer), achievement, and understanding (Sowell). Conceptual knowledge originates in the inventive activities of the learner through actions on objects rather than from sensory impression or social transmission derived from teachers and parents (Piaget)." (MacKenzie)

Many of the teaching strategies in this document suggest the use of manipulatives to demonstrate and involve students in the process of developing their own conceptual understanding and mental math skills. The use of manipulatives also serves as an effective method of communication for students. In addition, students will be more likely to retain information and ideas when involved actively in their own learning through the use of these tools in their math classes.

I hear and I forget
I see and I remember
I do and I understand.
—Ancient Chinese Proverb

Mental Math

Grade 8 Mathematics

S–2

Sample Strategy

Place-value partitioning: Adding from left to right

Starting from the right and working left can be an efficient method to solve addition questions mentally …

EXAMPLE 1

$$\begin{array}{r} 46 \\ +\ 38 \\ \hline \end{array}$$

$6 + 8 = 14$

$40 + 30 = 70$

$70 + 14 = \mathbf{84}$

…but often, a more efficient method for mentally adding is to start from the left side of the equation.

EXAMPLE 2

$$\begin{array}{r} 46 \\ +\ 38 \\ \hline \end{array}$$

$40 + 30 = 70$

$6 + 8 = 14$

$70 + 14 = \mathbf{84}$

EXAMPLE 3

$$\begin{array}{r} 25.6 \\ +\ 13.7 \\ \hline \end{array}$$

$20 + 10 = 30$

$5 + 3 = 8$

$30 + 8 = 38$

$0.6 + 0.7 = 1.3$

$38 + 1.3 = \mathbf{39.3}$

The benefit of adding from left to right is that you do not need to store as many numbers in your head as with the standard algorithm. You never have to regroup (carry), and place value is always maintained.

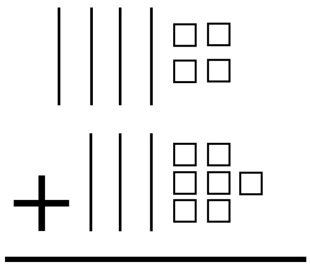
Teaching Strategies for Sample Strategy S–2

S–2

Place-value partitioning: Adding from left to right

Use Base-10 blocks to show that adding from left to right can be an efficient method of performing mental math.

Example 1: 44 + 37



Show 44 + 37 with Base-10 blocks.

- show adding from left to right by adding groups of tens first, then regrouping units into tens if possible, and finally counting the remaining units

  40 + 30 + 10 + 1 = 81

- compare this to trying to add the total, starting with the units first

  4 + 7 = 11

  11 + 70 = 81

- reflect: Is it easier to add tens first, units first, or does it really matter?
- reflect: Is it quicker to report using one of the methods?

Students may find that the first approach is more efficient. By starting with the largest place value and working progressively smaller, you can avoid mentally sorting out tens from ones. You can also report the answer as you work through the question.

Example 2: Working left to right, 333 + 156

Add the hundreds first and right away you can report, "*four hundred*...."

Add the tens and report, "eighty-...."

Add the units and report "nine."

Working memory does not need to be used to store the value of each digit. As you continue working through the question, you report the answer at the same time. When these steps involve regrouping (carrying), a simple step of substituting the previous number corrects the answer.

When working from right to left, each calculated digit needs to be stored in working memory. An additional final step is necessary to reverse the order of all of these digits. The conceptual importance of place value can be lost with this approach.

Working right to left, for example, with 333 + 156:

3 + 6 = 9 (Store this in your memory.)

3 + 5 = 8 (Store this in your memory. Notice how place value loses all significance.)

3 + 1 = 4 (Store this in your memory.)

Reverse all the digits: 9, 8, 4 becomes 4, 8, 9.

Now reassign a place value to all of the digits and report.

4, 8, 9 becomes *four hundred eighty-nine*.

With paper and pencil, the standard algorithm is generally more efficient because any regrouping/carrying is addressed immediately, and erasing is not needed. With mental computation, however, students should realize, and be encouraged to implement, working from left to right as a more efficient process.

Mental Math

Grade 8 Mathematics

S–3

Sample Strategy

Break down numbers and add their parts

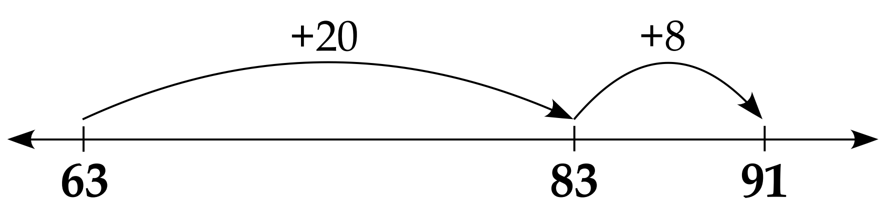
Here's another method of doing addition in your head:

EXAMPLE 1

$$\begin{array}{r} 63 \\ +\ 28 \\ \hline \end{array}$$

63 + 20 = 83

83 + 8 = **91**



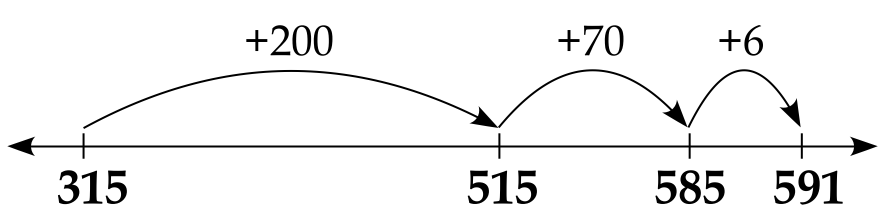
Break the numbers up and add the parts in the order that works best for you.

EXAMPLE 2

$$\begin{array}{r} 315 \\ +\ 276 \\ \hline \end{array}$$

315 + 200 + 70 + 6

= **591**



When you break down numbers and add their parts, you never have to regroup (carry) and place value is always maintained.

Mental Math: Grade 8 Mathematics

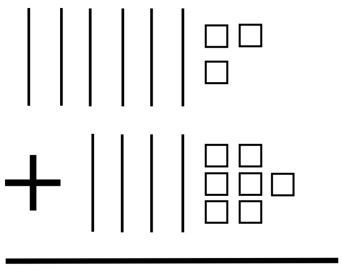
Teaching Strategies for Sample Strategy S–3

S–3

Break down numbers and add their parts

Use Base-10 blocks to show that numbers can be broken apart and added in many different ways.

Example 1: 63 + 47



Show 63 + 47 with Base-10 blocks.

Reflect: How many different ways can the total be found?

63 + 7 = 70

70 + 40 = 110

60 + 40 = 100

100 + 3 + 7 = 110

3 + 7 + 60 + 40 = 110

etc.

Reflect: Which ways are most efficient or easiest to use? Why?

Example 2: Using a number line to show 12.3 + 48.7

Have students use a metre stick or tape measure as a number line.

Show 12.3 cm + 48.7 cm

Reflect: How many different ways can you add these numbers to get the answer?

12.3 cm + 40 = 52.3 cm 52.3 + 8 = 60.3 cm 60.3 + 0.7 = 61 cm	12 cm + 48 = 60 cm 60 cm + 0.3 + 0.7 = 61 cm
12.3 cm + 0.7 = 13 cm 13 + 40 = 53 cm 53 + 8 = 61 cm	0.7 cm + 0.3 = 1 cm 1 + 2 + 8 = 11 cm 11 + 10 + 40 = 61 cm

Reflect: Which ways are most efficient or easiest to use? Why?

Students should come to some conclusions, such as that starting with compatible (friendly) numbers is sometimes easiest, and that numbers can be broken apart and worked with in many ways. Doing mental computation from left to right is generally one of the most efficient methods of adding numbers mentally, but when friendly numbers are present, starting wherever they occur may be even more efficient.

Mental Math

Grade 8 Mathematics

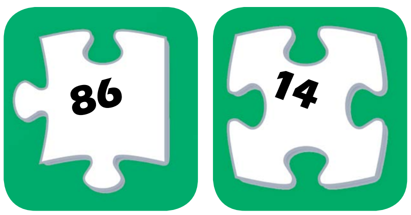
S–4

Sample Strategy

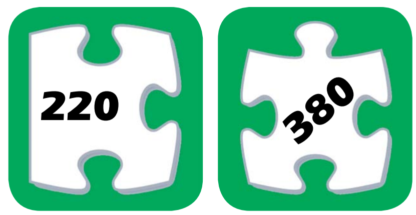
Finding compatible numbers (also known as friendly numbers)

Compatible numbers are pairs of numbers that are easy to add in your head.

The following are examples of compatible numbers:



The sum equals 100.



The sum equals 600.

EXAMPLE 1

Find the pairs of compatible numbers that add up to 300.

140	85	160
118	217	73
215	182	83

140 + 160

118 + 182

215 + 85

EXAMPLE 2

Find the pairs of compatible numbers that add up to 800.

250	175	567
333	440	467
625	550	360

250 + 550

333 + 467

625 + 175

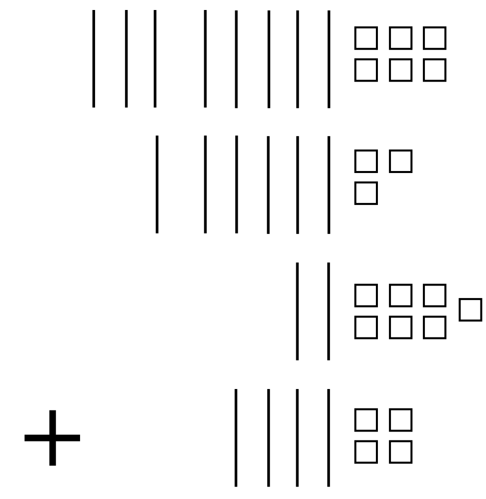
Teaching Strategies for Sample Strategy S–4

S–4

Finding compatible numbers (also known as friendly numbers)

Use Base-10 blocks to show that grouping compatible numbers makes them easier to add.

Example 1: 86 + 63 + 27 + 44



Model 86 + 63 + 27 + 44, using Base-10 blocks.

Reflect: What is the most efficient method of finding the total?

- Adding all of the units and all of the tens separately and then adding them both together (similar to the algorithm)
- Counting all
- Adding tens and counting up
- Regrouping in some other method

Encourage students to look for groups of friendly numbers.

80 + 20 = 100

60 + 40 = 100

6 + 4 = 10

3 + 7 = 10

100 + 100 + 10 + 10 = 220

An algorithmic approach to an addition question with this many addends is very difficult to do mentally. When working with more than two addends, looking first for compatible numbers is often the most efficient strategy. Even if there are some addends that are not compatible, mentally computing visible, friendly numbers reduces the number of steps necessary to solve an addition question of this type.

Try the following, searching for compatible numbers:

38 + 62 + 1	136 + 893 + 7 + 64	61 + 76 + 239 + 824	8.09 + 7.91 + 2
$\frac{1}{4} + \frac{1}{2} + \frac{3}{4}$	43% + 2% + 37%	\$4.23 + \$6.55 + \$3.45	2.3 km + 700 m

Mental Math

Grade 8 Mathematics

S–5

Sample Strategy

Compensation for adding and subtracting, using compatible numbers

Sometimes it is easier to do addition and subtraction in your head by creating your own compatible numbers and adjusting the total.

EXAMPLE 1

$$\begin{array}{r} 650 \\ +\ 375 \\ \hline \end{array}$$

$650 + 350 = 1000$

$1000 + 25 = \mathbf{1025}$

EXAMPLE 2

$$\begin{array}{r} 1250 \\ -\ \ 753 \\ \hline \end{array}$$

$1250 - 750 = 500$

$500 - 3 = \mathbf{497}$

EXAMPLE 3

$$\begin{array}{r} 53 \\ +\ 39 \\ \hline \end{array}$$

$39 + 1 = 40$

$53 + 40 = 93$

$93 - 1 = \mathbf{92}$

EXAMPLE 4

$$6\frac{1}{5} - 2\frac{4}{5}$$

$$6\frac{1}{5} - \left(2\frac{4}{5} + \frac{1}{5}\right) = 6\frac{1}{5} - 3$$

$$= 3\frac{1}{5}$$

$$3\frac{1}{5} + \frac{1}{5} = \mathbf{3\frac{2}{5}}$$

Mental Math: Grade 8 Mathematics

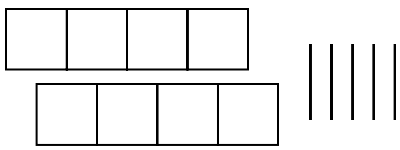
Teaching Strategies for Sample Strategy S–5

S–5

Compensation for adding and subtracting, using compatible numbers

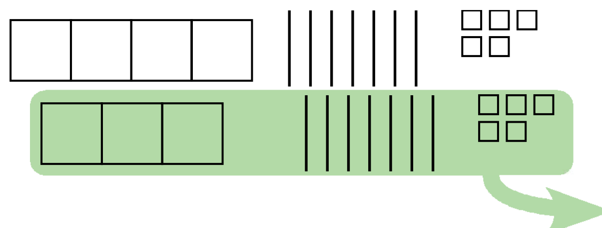
Use Base-10 blocks to demonstrate the extra work involved in regrouping, which can be eliminated by using compatible (friendly) numbers.

Example 1: Have students model 850 − 375 using Base-10 blocks

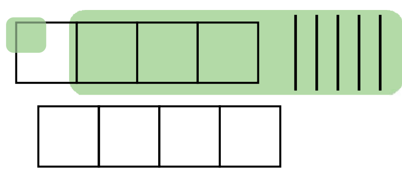


This will require several steps to complete. A ten will need to be broken down into ones, and a hundred will need to be broken into tens.

After regrouping the 850, pulling 375 units away will leave 4 hundreds, 7 tens, and 5 units = 475.



Ask students to look at the question of 850 − 375 as 850 − 350 (a compatible number) − 25.



850 − 375 = 850 − 350 − 25

850 − 350 = 500

500 − 25 = 475

By creating friendly numbers, the borrowing/regrouping process is eliminated, simplifying the process.

Doing this work mentally eliminates the need to regroup (or borrow) when doing subtraction equations. Using compatible numbers also maintains place value.

For example: 823 − 730 =

Using the method of creating compatible numbers:

823 − 730 = 823 − 723 − 7
823 − 723 = 100
100 − 7 = *ninety-three*

Using the algorithmic process mentally:

3 − 0 = 3
2 − 3 doesn't work, so regroup tens (borrow)
12 − 3 = 9
8 changed to a 7, so 7 − 7 = 0
Reverse the numbers of 3, 9, 0 to 093 and reassign place value = *ninety-three*

It should become apparent to students that, although the algorithmic method works well on paper, mentally it is a much more cumbersome process than using compatible numbers.

Mental Math

Grade 8 Mathematics

S–6

Sample Strategy

Subtracting, balancing both elements

When you add the same number to the two elements of a subtraction question, the difference between the two does not change.

EXAMPLE 1

$$\begin{array}{r} 4.32 \\ -\ 1.95 \\ \hline \end{array}$$

$4.32 + 0.05 = 4.37$

$1.95 + 0.05 = 2$

$4.37 - 2 = \mathbf{2.37}$

EXAMPLE 2

$$\begin{array}{r} 96.3 \\ -\ 30.1 \\ \hline \end{array}$$

$30.1 + (-0.1) = 30$

$96.3 + (-0.1) = 96.2$

$96.2 - 30 = \mathbf{66.2}$

EXAMPLE 3

$$6\frac{1}{3} - 1\frac{2}{3}$$

$1\frac{2}{3} + \frac{1}{3} = 2$

$6\frac{1}{3} + \frac{1}{3} = 6\frac{2}{3}$

$6\frac{2}{3} - 2 = \mathbf{4\frac{2}{3}}$

Subtracting, balancing both elements

Use volume to demonstrate that adding or subtracting the same number from both elements in a subtraction question will always result in the same difference.

Required materials: 2 measuring cups
a water source
measuring spoons

1. Fill one measuring cup up to the 500 mL mark and the other to the 750 mL mark. Ask students to find the difference between the two in mL.
2. Add 100 mL to both cups and ask students to find the difference between the two.
3. Add 125 mL more to both cups and ask students to again find the difference between the two.
4. Continue adding equivalent amounts of water to each measuring cup until students see clearly that the difference will remain constant as long as an equal amount is added to both measuring cups.
5. Remove 150 mL from each measuring cup and ask students to find the difference.
6. Remove 50 mL from each measuring cup and ask students to find the difference.
7. Continue removing equivalent amounts of water from each measuring cup until students realize that the difference will remain constant as long as an equal amount is subtracted from both measuring cups.

Extension: Try the same activity, using the imperial system and fractions of cups.

Have students develop a rule from this activity. It should be similar to the following:

When you add the same number to the two elements of a subtraction question, the difference between the two does not change.

$$(a - b) = (a + n) - (b + n)$$

and

$$(a - b) = (a - n) - (b - n)$$

Mental Math

Grade 8 Mathematics

S–7

Sample Strategy

Subtract starting from the left: Place-value positioning

Mentally subtracting when regrouping (borrowing) is involved takes a small additional step, but it is a very effective and useful strategy.

EXAMPLE 1

$$\begin{array}{r} 468 \\ -\ 323 \\ \hline \end{array}$$

Scan the question. No regrouping is needed.

400 – 300 = 100

60 – 20 = 40

8 – 3 = 5

100 + 40 + 5 = **145**

EXAMPLE 2

$$\begin{array}{r} 9514 \\ -\ 6233 \\ \hline \end{array}$$

Scan the question. Regrouping will be needed for the tens place. Adjust the hundreds place to reflect this.

9000 – 6000 = 3000

400 – 200 = 200

110 – 30 = 80

4 – 3 = 1

3000 + 200 + 80 + 1 = **3281**

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–7

S–7

Subtract starting from the left: Place-value positioning

Explain various mental math strategies to demonstrate that starting subtraction from the left side is efficient.

Have students mentally solve the following question using a Mental Math Student Communication Template and the method described in the Reproducible Sheets section. (Notice that no regrouping is involved in this specific question and that there are many other possible methods of mentally solving this equation, such as compatible numbers or compensation.)

Question: 974 − 343	Answer: 631
Method used to calculate mentally (a right-to-left, standard algorithm approach): 1. 4 − 3 = 1 (mentally store) 2. 7 − 4 = 3 (mentally store) 3. 9 − 3 = 6 (mentally store) 4. Mentally retrieve and reverse the order from 1, 3, 6 to 6, 3, 1 and reassign place value 5. Say the answer as "**six hundred thirty-one**"	A method used by someone else (a left-to-right approach): 1. 900 − 300 = say, "***six hundred...***" 2. 70 − 40 = say, "***thirty...***" 3. 4 − 3 = say, "***one.***"
Preferred method and reason: Second method takes fewer steps and less mental storing. There is less chance for error.	

Getting the correct answer is not as important as the process used to solve this question. Many students may get the wrong answer by following the algorithmic approach because the many steps involving mental storing and retrieval can become difficult to manage. Guide students to see that, although the algorithmic approach is usually the most effective method when working with pencil and paper, it is not often an effective mental math strategy. Also note that the left-to-right approach maintains the importance of place value.

Try the following question, which involves regrouping:

Question: 814 − 78	Answer: 736
Method used to calculate mentally (right to left): 1. 4 − 8 requires regrouping 2. 14 − 8 = 6 (store mentally) 3. Recall that 1 was changed to 0. Regroup. 4. 10 − 7 = 3 (store mentally) 5. Recall that 8 was changed to 7. 7 − 0 = 7 6. Mentally retrieve and reverse the order from 6, 3, 7 to 736 7. Say the answer as, "**seven hundred thirty-six**."	A method used by someone else (left to right): 1. 800 − 0 = 800 (store mentally) 2. 10 − 70 = problem! Regroup 800 to 700 and say, "***seven hundred...***" 3. 110 − 70 = 40 (store mentally) 4. 4 − 8 = problem! Regroup 40 to 30 and say," ***thirty...***" 5. 14 − 8 = say, "***six.***"
Preferred method and reason: The second method is more efficient in this case. It also maintains place value throughout. This method would not be as efficient with paper and pencil, however, because there would be a lot of erasing.	

Mental Math

Grade 8 Mathematics

S–8

Sample Strategy

Subtracting one part at a time: Place-value positioning

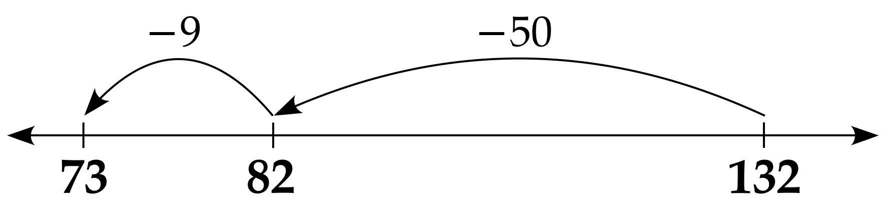
When solving a subtraction question that requires regrouping (borrowing), try subtracting one part at a time.

EXAMPLE 1

$$\begin{array}{r} 132 \\ -\ \ 59 \\ \hline \end{array}$$

$132 - 50 = 82$

$82 - 9 = \mathbf{73}$

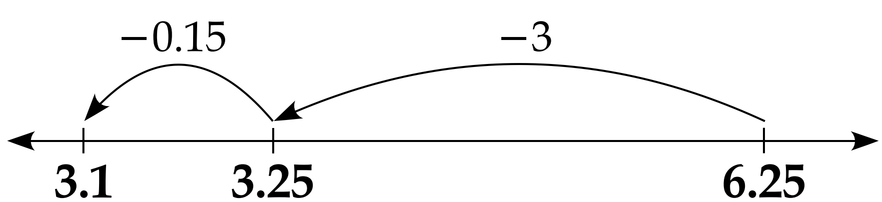


EXAMPLE 2

$$\begin{array}{r} 6.25 \\ -\ 3.15 \\ \hline \end{array}$$

$6.25 - 3 = 3.25$

$3.25 - 0.15 = \mathbf{3.1}$



Subtracting one part at a time: Place-value positioning

Use number lines to demonstrate the effectiveness of counting-back subtraction strategies when used with breaking up numbers into their smaller parts.

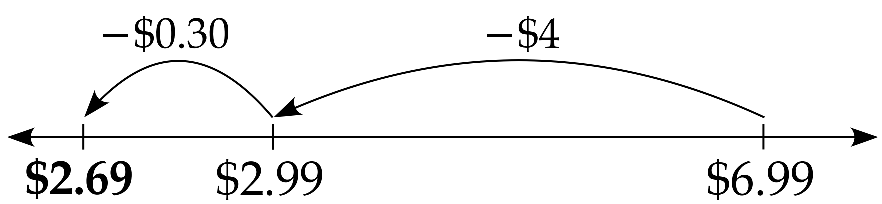
Have students find 29 cm − 15 cm using a ruler, and explain their process to a partner.

Share possible strategies with the class:

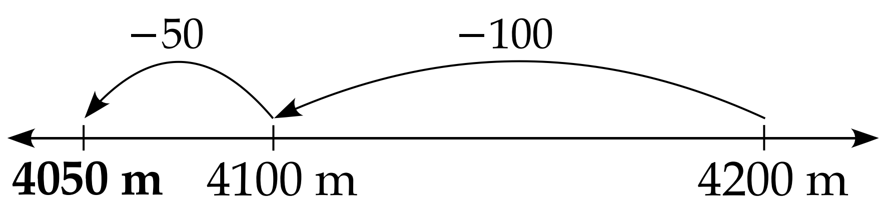
- Some may have used a strategy of counting back, starting with 29 and counting back by ones 15 times. This may have worked with this situation, but for larger numbers or decimals it would generally be inefficient.
- Some may have used a strategy of compensation, changing 29 to 30, subtracting 15, then subtracting 1 more to compensate. This is an efficient strategy for subtraction.
- Some may have seen the difference immediately as 14, or thought algebraically 15 + x = 29. This may have been possible for this situation, but not likely possible for decimals or larger numbers.
- Subtracting 1 part at a time is an effective strategy. Start at 29, subtract 10 to get to 19, and subtract 5 more to get to 14 cm. Breaking up the subtrahend will work in all situations.

Try drawing number lines and breaking up the subtrahend for the following questions. Note that number lines in this case do not have to be to scale.

\$6.99 − \$4.30



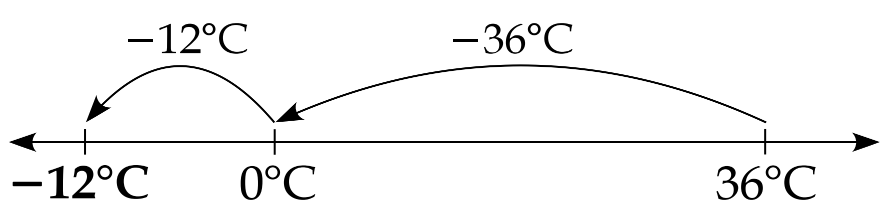
4.2 km − 150 m



In some situations, students may find that breaking up the subtrahend into other combinations of numbers may become even more efficient:

36°C − 48°C

36°C − 48°C = 36°C − (36°C + 12°C)
= 36°C − 36°C − 12°C



Mental Math

Grade 8 Mathematics

S–9

Sample Strategy

Subtraction: Thinking integers

When working on a subtraction question that involves regrouping (borrowing), a possible method is to make use of the way integers subtract.

EXAMPLE 1

$$\begin{array}{r} 243 \\ -\ 67 \\ \hline \end{array}$$

$200 - 0 = 200$

$40 - 60 = -20$

$3 - 7 = -4$

$200 - 20 - 4 = \mathbf{176}$

EXAMPLE 2

$$\begin{array}{r} \$9.95 \\ -\ \$6.87 \\ \hline \end{array}$$

$900¢ - 600¢ = 300¢$

$90¢ - 80¢ = +10¢$

$5¢ - 7¢ = -2¢$

$300¢ + 10¢ - 2¢ = \mathbf{\$3.08}$

EXAMPLE 3

$$34\frac{1}{4} - 16\frac{3}{4}$$

$30 - 10 = 20$

$4 - 6 = -2$

$\frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = -\frac{1}{2}$

$20 - 2 - \frac{1}{2} = \mathbf{17\frac{1}{2}}$

Working from left to right allows place value to be maintained.

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–9

S–9

Subtraction: Thinking integers

Using a method such as this requires that students have a very strong ability to work with negative integers, as well as a strong conceptual understanding of integers and the subtraction process. Teaching and encouraging the use of this strategy should only be done after students have a thorough understanding of both.

Potential sources of confusion for students:

- Students who are not entirely proficient at using the standard algorithm may get parts of this process confused with the standard algorithm and end up with misconceptions of both processes.
- Students in Early Years have often developed a misconception that a number cannot be subtracted from a number that is smaller than it. This misconception may have been encouraged through the use of the standard algorithm, which requires regrouping when a number is subtracted from a smaller number. Students with this misconception do not have a strong understanding of integers, and this method would not be appropriate for them.

Have students consider and discuss this situation through group discussion.

5	4	.1	3	
−	9	.7	4	
50	−5	−0.6	−0.01	**= 44.39**

Why does this method work?

How does this method work?

Does this method always work? Explain and give examples. If you find examples where this method doesn't work, use calculators to double-check.

Is this method something you would use? Why or why not?

The benefits of mentally using this thinking integer method are that work can be done from left to right and regrouping is entirely avoided. By avoiding regrouping, there are fewer numbers that students have to keep in their working memory and fewer opportunities for error. Place value is also maintained throughout this process.

Mental Math

Grade 8 Mathematics

S–10

Sample Strategy

Subtraction: Using addition to subtract/thinking addition

One method of solving subtraction questions is to reframe them in your mind as addition questions.

EXAMPLE 1

$$\begin{array}{r} 764 \\ -\ 698 \\ \hline \end{array}$$

$698 + \underline{\qquad} = 764$

$698 + \underline{\ \ 2\ \ } = 700$

$700 + \underline{\ 64\ } = 764$

$2 + 64 = \mathbf{66}$

EXAMPLE 2

$$\begin{array}{r} \$29.95 \\ -\ \$15.34 \\ \hline \end{array}$$

$\$15.34 + \underline{\qquad\quad} = \$29.95$

$\$15.34 + \underline{\$14.00} = \$29.34$

$\$29.34 + \underline{\ \$0.61\ } = \$29.95$

$\$14.00 + \$0.61 = \mathbf{\$14.61}$

EXAMPLE 3

$$7\frac{5}{8} - 4\frac{7}{8}$$

$4\frac{7}{8} + \underline{\qquad} = 7\frac{5}{8}$

$4\frac{7}{8} + \underline{\ \frac{1}{8}\ } = 5$

$5 + \underline{\ 2\frac{5}{8}\ } = 7\frac{5}{8}$

$2\frac{5}{8} + \frac{1}{8} = 2\frac{6}{8} = \mathbf{2\frac{3}{4}}$

Teaching Strategies for Sample Strategy S–10

S–10

Subtraction: Using addition to subtract/thinking addition

Thinking addition is likely one of the first ways most of your students learned to subtract single-digit numbers. Relate subtraction to the opposite operation of addition.

Show the opposite relationship of addition and subtraction using manipulatives and part-part-whole relationships.

Example 1

Show $255 - 55 = 200$ with Base-10 blocks.

255 is the whole, 55 is one of its parts, and 200 is another one of its parts

Have students write out as many pairs of equations as they can to represent this:

$255 - 55 = 200 \Rightarrow 200 + 55 = 255$

$255 - 200 = 55 \Rightarrow 55 + 200 = 255$

$100 + 100 + 50 + 5 = 255 \Rightarrow 255 - 5 - 50 - 100 = 100$

$255 - 5 - 5(10) = 200 \quad \Rightarrow 200 + 5(10) + 5 = 255$

etc.

Example 2

Practise making change with money.

$20.00 paid for a $16.75 T-shirt

$20.00 − $16.75 = **$3.25**

Think addition:

$16.75 + $3 = $19.75

$19.75 + $0.25 = $20.00

$3 + $0.25 = **$3.25**

Example 3

Show $3\frac{1}{6} - 1\frac{5}{6}$ with pattern blocks.

$3\frac{1}{6} - 1\frac{5}{6} = 1\frac{2}{6} = 1\frac{1}{3}$

Think addition:

$1\frac{5}{6} + \underline{\qquad} = 3\frac{1}{6}$

$1\frac{5}{6} + \underline{\;\frac{1}{6}\;} = 2$

$2 + \underline{\;1\frac{1}{6}\;} = 3\frac{1}{6}$

$1\frac{1}{6} + \frac{1}{6} = 1\frac{2}{6} = \mathbf{1\frac{1}{3}}$

Reaffirm for students that subtraction, in every possible situation, can be viewed as the opposite process to addition.

Mental Math

Grade 8 Mathematics

S–11

Sample Strategy

Division: Using multiplication to divide/thinking multiplication

One method of solving division questions is to reframe them in your mind as multiplication questions.

EXAMPLE 1 $444 \div 11.1$

$11.1 \times \underline{\qquad} = 444$

$11.1 \times \underline{\quad 4 \quad} = 44.4$

So, $11.1 \times \underline{\quad 40 \quad} = 444$

$444 \div 11.1 = \mathbf{40}$

EXAMPLE 2 $3 \div \frac{1}{2}$

$\frac{1}{2} \times \underline{\qquad} = 3$

$\frac{1}{2} \times \underline{\quad 6 \quad} = 3$

$3 \div \frac{1}{2} = \mathbf{6}$

EXAMPLE 3 $120 \div 0$

$0 \times \underline{\qquad} = 120$

No number will fit this!

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–11

S–11

Division: Using multiplication to divide/thinking multiplication

- Recall of the multiplication and related division facts up to 5×5 is expected by the end of Grade 4.
- Recall of multiplication facts to 81 and related division facts is expected by the end of Grade 5.
- Students are no longer expected to develop recall to 12×12 at any grade level, but are expected to have multiple strategies to perform this type of double-digit multiplication mentally.

Students should be aware of division's inverse relationship to multiplication.

Multiplication facts to 81

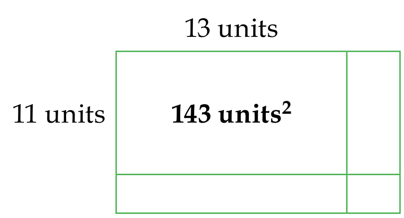
End of grade expectations:
Grade 4 and Grade 5

0 x 0	1 x 0	2 x 0	3 x 0	4 x 0	5 x 0	6 x 0	7 x 0	8 x 0	9 x 0
0 x 1	1 x 1	2 x 1	3 x 1	4 x 1	5 x 1	6 x 1	7 x 1	8 x 1	9 x 1
0 x 2	1 x 2	2 x 2	3 x 2	4 x 2	5 x 2	6 x 2	7 x 2	8 x 2	9 x 2
0 x 3	1 x 3	2 x 3	3 x 3	4 x 3	5 x 3	6 x 3	7 x 3	8 x 3	9 x 3
0 x 4	1 x 4	2 x 4	3 x 4	4 x 4	5 x 4	6 x 4	7 x 4	8 x 4	9 x 4
0 x 5	1 x 5	2 x 5	3 x 5	4 x 5	5 x 5	6 x 5	7 x 5	8 x 5	9 x 5
0 x 6	1 x 6	2 x 6	3 x 6	4 x 6	5 x 6	6 x 6	7 x 6	8 x 6	9 x 6
0 x 7	1 x 7	2 x 7	3 x 7	4 x 7	5 x 7	6 x 7	7 x 7	8 x 7	9 x 7
0 x 8	1 x 8	2 x 8	3 x 8	4 x 8	5 x 8	6 x 8	7 x 8	8 x 8	9 x 8
0 x 9	1 x 9	2 x 9	3 x 9	4 x 9	5 x 9	6 x 9	7 x 9	8 x 9	9 x 9

Having students practise solving equations involving multiplication and related division facts can be one way to build these basic skills.

$30 \div 6 = y$	$56 \div 7 = m$	$42 \div 7 = f$	$12 \div \frac{1}{4} = d$
$6 \times y = 30$	$7 \times m = 56$	$7 \times f = 42$	$0.25 \times d = 12$
$y \times 6 = 30$	$m \times 7 = 56$	$f \times 7 = 42$	$d \times 25\% = 12$
$30 \div y = 6$	$56 \div m = 7$	$42 \div f = 7$	$\frac{12}{d} = 0.25$

Algebra tiles can also be effectively used to show that there is an inverse relationship between the area of an array and the length of one of its sides.



$13 \times 11 = 143$
$11 \times 13 = 143$
$143 \div 11 = 13$
$143 \div 13 = 11$

Mental Math

Grade 8 Mathematics

S–12

Sample Strategy

Place-value partitioning: Multiplication and division

It can often be easier to multiply or divide in your head when you break down a number and start from the left.

EXAMPLE 1

$$\begin{array}{r} 635 \\ \times \quad 4 \\ \hline \end{array}$$

$600 \times 4 = 2400$

$30 \times 4 = 120$

$5 \times 4 = 20$

$3400 + 120 + 12 = \mathbf{2540}$

EXAMPLE 2

$$1452 \div 4$$

$1000 \div 4 = 250$

$400 \div 4 = 100$

$52 = 40 + 12$

$40 \div 4 = 10$

$12 \div 4 = 3$

$250 + 100 + 10 + 3 = \mathbf{363}$

EXAMPLE 3

$$2 \times 4\frac{3}{8}$$

$2 \times 4 = 8$

$2 \times \frac{3}{8} = \frac{6}{8}$

$8 + \frac{6}{8} = 8\frac{6}{8} = \mathbf{8\frac{3}{4}}$

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–12

S–12

Place-value partitioning: Multiplication and division

Explain various mental math strategies to demonstrate that place-value partitioning for multiplication and division is often most efficient.

Have students mentally solve the following question using a Mental Math Student Communication Template and the method described in the Reproducible Sheets section.

Question: 912 × 7	Answer: 6384
Method used to calculate mentally (a right-to-left, standard algorithmic approach): 1. 2 × 7 = 14 (mentally store the 4, regroup the 10 as 1) 2. 1 × 7 + 1 = 8 (mentally store) 3. 9 × 7 = 63 (mentally store) 4. Mentally retrieve and reverse the order from 4, 8, 63 to 63, 8, 4 and reassign place value 5. Say the answer as "**six thousand three hundred eighty-four**."	A method used by someone else (a left-to-right approach): 1. Scan, 2 × 7 is the only column that will require regrouping. 2. 900 × 7 = say, "***six thousand three hundred...***" 3. 10 × 7 = 70 (mentally store) 4. 2 × 7 = 14 (mentally store) 5. 70 + 14 = say, "***eighty-four.***"
Preferred method and reason: Second method takes fewer steps and less mental storing. There is less chance for error.	

Getting the correct answer is not as important as the process used to solve this question. Many students may get the wrong answer by following the algorithmic approach because the many steps involving mental storing and retrieval can become difficult to manage. Guide students to see that, although the algorithmic approach is usually the most effective method when working with pencil and paper, it is not often an effective mental math strategy.

Try the following question, which involves dividing:

Question: 2052 ÷ 6	Answer: 342
Method used to calculate mentally (right to left): 1. 6 does not go into 2* 2. 20 ÷ 6 = 3 (remember this) 3. 3 × 6 = 18 4. 20 − 18 = 2 5. Drop down the 5 to make 25. 6. 25 ÷ 6 is 4 (remember this) 7. 4 × 6 = 24 8. 25 − 24 = 1, bring down the 2 to make 12. 9. 12 ÷ 6 = 2 (remember this) 10. Reassign place value, and say the answer as, "**three hundred forty-two**."	A method used by someone else (left to right): 1. 2052 = 1800 + 240 + 12 (store these mentally) 2. 1800 ÷ 6 = say, "***three hundred...***" 3. 240 ÷ 6 = say, "***forty-...***" 4. 12 ÷ 6 = say, "***two.***"
Preferred method and reason: The second method is more efficient in this case. It also maintains place value throughout.	

*** Vocabulary like this can be problematic. Students may come to believe that division can only take place with smaller numbers fitting into larger ones. $2 \div 6 = \frac{1}{3}$, but for the algorithm, we treat this as invalid.**

Mental Math

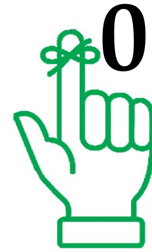
Grade 8 Mathematics

S–13

Sample Strategy

Annexing zeros and working with powers of ten/cutting and pasting zeros

In multiplication and division, annexing zeros allows for quick mental computation of whole numbers that are multiples of powers of ten.



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EXAMPLE 1

$$\begin{array}{r} 6000 \\ \times \quad 30 \\ \hline \end{array}$$

$6\cancel{000} \times 3\cancel{0} = \mathbf{180\ 000}$

EXAMPLE 2

$17\ 500 \div 25$

$17\ 5\cancel{00} \div 25 = \mathbf{7\underline{00}}$

EXAMPLE 3

$88\ 000 \div 400$

Cancel out zeros where possible for division questions

$88\ 0\cancel{00} \div 4\cancel{00} = 880 \div 4 =$

$88\cancel{0} \div 4 = 22\underline{\mathbf{0}}$

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–13

S–13

Multiplying and dividing by a power of 10

Have students mentally solve the following questions using a Mental Math Student Communication Template and the method described in the Reproducible Sheets section.

Annexing zeros algorithm for multiplication:

1. Cut all the trailing zeros for numbers being multiplied.
2. Multiply the remaining numbers.
3. Paste all the zeros back.

Question: 6000×30	Answer: 180 000
Method used to calculate mentally (using annexing zeros): $6\cancel{000} \times 3\cancel{0} = 18\underline{\mathbf{0\ 000}}$	A method used by someone else: $6000 \times 30 = (6 \times 1000) \times (3 \times 10)$ $= 6 \times 3 \times 1000 \times 10$ $= 18 \times 10\ 000$ $= 180\ 000$
Preferred method and reason: Annexing zeros method takes only one step.	

Annexing zeros algorithm for division:

1. Permanently cancel out zeros from both the dividend and divisor where possible.
2. Cut the remaining zeros from either the dividend or divisor.
3. Divide the remaining numbers.
4. Paste the zeros from the second step.

Question: $6300 \div 90$	Answer: 70
Method used to calculate mentally (using annexing zeros): $630\cancel{0} \div 9\cancel{0} = 630 \div 9$ $63\cancel{0} \div 9 = 7\underline{\mathbf{0}}$	A method used by someone else: $6300 \div 90 = (630 \times 10) \div (9 \times 10)$ $= (630 \div 9) \times (10 \div 10)$ $= 630 \div 9 \times 1$ $= 630 \div 9$ $= 70$
Preferred method and reason: Annexing zeros method takes fewer steps.	

Mental Math

Grade 8 Mathematics

S–14

Sample Strategy

Imagining a moving decimal point: Multiplication and division

Although the decimal point never really moves, imagining it as shifting can simplify the process of mentally multiplying and dividing by powers of ten.

Thousands	Hundreds	Tens	Ones	Decimal	Tenths	Hundredths	Thousandths
				•			

× 0.001	× 0.01	× 0.1	× 1	× 10	× 100
÷ 1000	÷ 100	÷ 10	÷ 1	÷ 0.1	÷ 0.01

EXAMPLE 1 6000×200

$6\cancel{000} \times 2\cancel{00} = 12$

shift decimal right one position for every zero in the question:

= **1 200 000**

EXAMPLE 2 $90\,000 \times 0.06$

$9\cancel{0\,000} \times \cancel{0.0}6 = 54$

shift decimal right four times and left twice:

= **5400**

EXAMPLE 3 $72 \div 0.09$

$72 \div \cancel{0.0}9 = 8$

shift decimal right twice:

= **800**

Imagining a moving decimal point: Multiplication and division

Show by induction that the decimal point can be seen as shifting for multiplication and division questions.

Have students answer the following series of questions and others similar to it, with or without calculators, noticing where the decimal points appear in the final answers. Have students develop rules:

40×1000	4×1000	0.4×1000	$4 \div 0.001$
40×100	4×100	0.4×100	$4 \div 0.01$
40×10	4×10	0.4×10	$4 \div 0.1$
40×1	4×1	0.4×1	$4 \div 1$
40×0.1	4×0.1	0.4×0.1	$4 \div 10$
40×0.01	4×0.01	0.4×0.01	$4 \div 100$
40×0.001	4×0.001	0.4×0.001	$4 \div 1000$

Rules developed may include the following:

For multiplying a number by a power of 10 greater than 1

Ignore the trailing zeros and multiply. Mentally shift the decimal place in your answer **right** one place for every trailing zero in the question.

For multiplying a number by a power of 10 less than 1

Ignore leading zeros and decimals and multiply. Mentally shift the decimal place in your answer **left** one place for every decimal place in the question.

For dividing a number by a power of 10 greater than 1

Ignore the trailing zeros and divide. Mentally shift the decimal place in your answer **left** one place for every trailing zero in the question.

For dividing a number by a power of 10 less than 1

Ignore leading zeros and decimals and divide. Mentally shift the decimal place in your answer **right** one place for every decimal place in the question.

Multiplying by 10 is the same as dividing by 0.1.

Multiplying by 100 is the same as dividing by 0.01.

Mental Math

Grade 8 Mathematics

Sample Strategy

Compensation: Multiplication and division

Add or subtract a group to a portion of the question in order to make it easier to solve. Compensate your final answer by performing the opposite operation on it.

EXAMPLE 1 $\$0.73 \times 4$

$\$0.73 + \$0.02 = \$0.75$

$\$0.75 \times 4 = \$3.00$

$\$3.00 - (\$0.02 \times 4) = \mathbf{\$2.92}$

EXAMPLE 2 $7992 \div 8$

$7992 + 1$ group of $8 = 8000$

$8000 \div 8 = 1000$

$1000 - 1 = \mathbf{999}$

EXAMPLE 3 $5\frac{7}{8} \times 4$

$6 \times 4 = 24$

$24 - \left(4 \times \frac{1}{8}\right) =$

$24 - \frac{1}{2} = \mathbf{23\frac{1}{2}}$

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–15

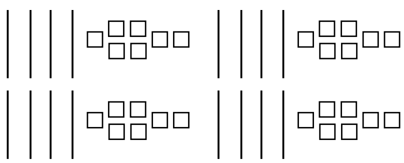
S–15

Compensation: Multiplication and division

Show students that using compensation strategies for multiplication and division simplifies mental computations.

Compensation with Multiplication

1. Demonstrate the question 47 × 4 with Base-10 blocks by showing 4 groups of 47.
2. Have students find the total using their own preferred method. They will get 188.

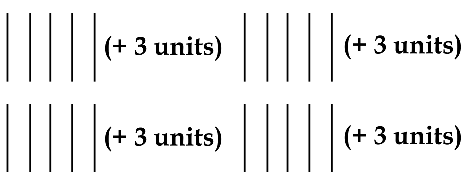


four groups of forty-seven
= 188 total

3. Show how rounding 47 to 50 makes this problem easier to solve. All that is required is that 3 be added to each of the four groups (3 × 4 = 12 in all).

   Students should be able to see fairly quickly that four groups of 50 is 200, and that this is easier to model and solve.

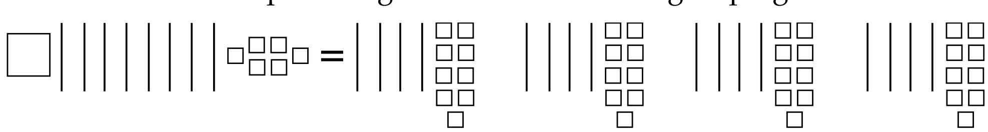
4. In order to do this, four extra groups of 3 had to be added, so in order to compensate for this, four groups of 3 need to be removed from the final answer. 200 − 12 = 188



= 200, with 12 extra units
200 − 12 = 188

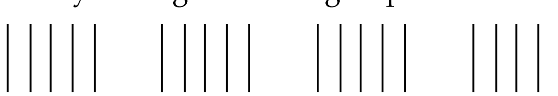
Compensation with Division

1. Model 196 ÷ 4 with Base-10 blocks, and have students show and solve the equation. Showing this with Base-10 blocks will require a significant amount of regrouping.



196 ÷ 4 = 49

2. Have students try adding one more group of 4 to 196 to try the same question.



200 ÷ 4 = 50

3. Have students finish the compensation process by mentally removing one unit from each group, and they should see that 196 ÷ 4 = 49.

Mental Math

Grade 8 Mathematics

S–16

Sample Strategy

Dividing: Balancing both elements

When you multiply the same number to the two elements of a division question, or divide both elements by the same number, the ratio between the two does not change and the answer always remains the same.

$$a \div b = a(c) \div b(c) \qquad a \div b = \frac{a}{c} \div \frac{b}{c}$$

EXAMPLE 1 $7.5 \div 1.5$

$7.5(2) \div 1.5(2) = 15 \div 3$

$= \mathbf{5}$

EXAMPLE 2 $240 \div 48$

$= (240 \div 6) \div (48 \div 6)$

$= 40 \div 8 = \mathbf{5}$

EXAMPLE 3 $6\frac{2}{3} \div 1\frac{2}{3}$

$= \frac{20}{3} \div \frac{5}{3}$

$= \left(\frac{20}{3} \times 3\right) \div \left(\frac{5}{3} \times 3\right)$

$= 20 \div 5 = \mathbf{4}$

Dividing: Balancing both elements

Show, using counters, that the ratio between elements of a division question remains the same when both elements are divided by the same number (except for zero).

- Have students show 36 ÷ 6 using counters. Students should have six distinct groups of 6.
- Using this arrangement, have students pile two counters from within the same group on top of each other. Their model will still represent 36 ÷ 6 = 6, but their piles of counters will be showing 18 ÷ 3 = 6 (eighteen piles, divided into groups of 3, equals six). They are modelling that

$$\begin{aligned} 36 \div 6 &= \frac{36}{2} \div \frac{6}{2} \\ &= 18 \div 3 \\ &= 6 \end{aligned}$$

- Have students regroup each of their piles so that each group now has two piles of 3, showing both 36 ÷ 6 = 6, and 12 ÷ 2 = 6 (twelve piles, divided into groups of 2, equals six). They will be modelling that

$$\begin{aligned} 36 \div 6 &= \frac{36}{3} \div \frac{6}{3} \\ &= 12 \div 2 \\ &= 6 \end{aligned}$$

- Have students try the same with 40 counters to show that

$$40 \div 4 = 20 \div 2 = 10 \div 1$$

Use $a \div b = a(c) \div b(c)$ to show why, when dividing two fractions, we can multiply by the reciprocal to get the same result.

$$\begin{aligned} &\frac{a}{b} \div \frac{c}{d} \\ &= \frac{a}{b}\left(\frac{d}{c}\right) \div \boxed{\frac{c}{d}\left(\frac{d}{c}\right)} \\ &= \frac{a}{b}\left(\frac{d}{c}\right) \div 1 \\ &= \frac{a}{b}\left(\frac{d}{c}\right) \end{aligned}$$

We multiply by the inverse because it will equal 1!

$$\boldsymbol{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}}$$

$$\begin{aligned} &\frac{1}{3} \div \frac{4}{7} \\ &= \frac{1}{3}\left(\frac{7}{4}\right) \div \frac{4}{7}\left(\frac{7}{4}\right) \\ &= \frac{1}{3}\left(\frac{7}{4}\right) \div 1 \\ &= \frac{1}{3}\left(\frac{7}{4}\right) \end{aligned}$$

$$\boldsymbol{\frac{1}{3} \div \frac{4}{7} = \frac{1}{3} \times \frac{7}{4}}$$

Mental Math

Grade 8 Mathematics

S–17

Sample Strategy

Doubling AND halving

In multiplication, doubling one factor and halving the other will give you the same result.

EXAMPLE 1 15×12

$15 \times 12 =$

$30 \times 6 = \mathbf{180}$

EXAMPLE 2 $12\frac{1}{2} \times 4$

$12\frac{1}{2} \times 4 =$

$25 \times 2 = \mathbf{50}$

Doubling OR halving

Using known facts, double them or halve them to determine the answer.

EXAMPLE 3 7×16

Think the double of 7×8

$7 \times 8 = 56$

$56 \times 2 = \mathbf{112}$

EXAMPLE 4 $164 \div 8$

Think repeated halving

$164 \div 2 = 82$

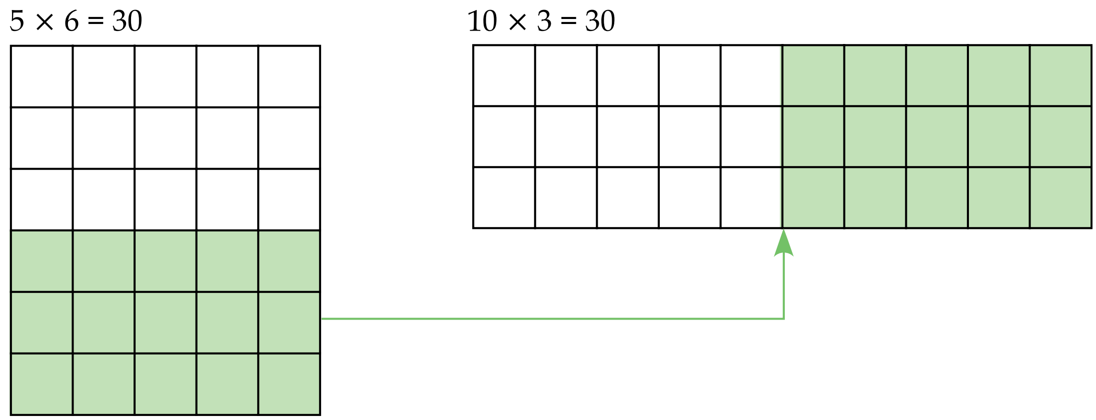
$82 \div 2 = 41$

$41 \div 2 = \mathbf{20\frac{1}{2}}$

Doubling and/or halving

Show using arrays (an area model) the effects of doubling and/or halving elements of a multiplication and division question.

Doubling AND Halving



- Have students, using grid paper, come up with other examples of where doubling and halving works well.
- Ask students to develop guidelines as to where this process might be effective in mental math situations. Students should eventually see that this method works best when working with at least one even number, or that when one element ends in a five, doubling it makes it a power of ten and it becomes easier to work with.
- A large selection of practice questions are available at: https://nzmaths.co.nz/sites/default/files/DoublingAndHalvingSheet.pdf.

Doubling OR Halving

- This process can be used to solidify basic multiplication and division facts.
- Encouraging students to think of doubles and halves when working with 6s and 8s is an effective step towards developing automaticity.

  6×3 is double 3×3

  6×4 is double 3×4 OR double 6×2

  6×6 is double 3×6 OR quadruple 3×3

  6×7 is double 3×7

  7×8 is double 7×4 OR quadruple 7×2

  $48 \div 12$ is half $48 \div 6$

  $24 \div 3$ is double $24 \div 6$

Mental Math

Grade 8 Mathematics

S–18

Sample Strategy

Properties and laws: Commutative property

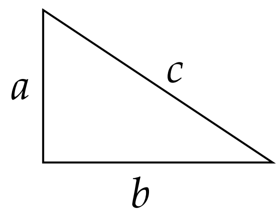
When adding or multiplying numbers together, switch them around in ways that make them easier to work with.

$a + b = b + a$

$a \times b = b \times a$

EXAMPLE 1

Find the hypotenuse.



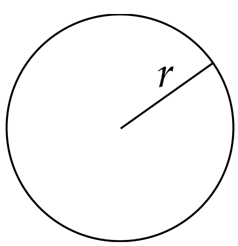
$a^2 + b^2 = c^2$

so

$b^2 + a^2 = c^2$

EXAMPLE 2

Find the diameter.



$d = 2r$

so

$r \times 2 = d$

EXAMPLE 3

Solve.

$$\frac{2}{3} \times \frac{6}{7}$$

$$\begin{aligned} \frac{2}{3} \times \frac{6}{7} &= \frac{6}{3} \times \frac{2}{7} \\ &= 2 \times \frac{2}{7} \\ &= \frac{4}{7} \end{aligned}$$

Mental Math

Grade 8 Mathematics

S–19

Sample Strategy

Properties and laws: Associative property

When adding or multiplying many numbers together, order them in ways that make them easier to work with.

$(a + b) + c = a + (b + c)$

$(a \times b) \times c = a \times (b \times c)$

EXAMPLE 1

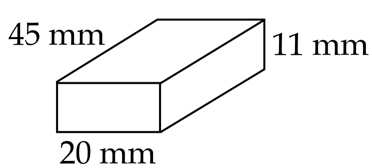
$$\begin{array}{r} 42 \\ 17 \\ 18 \\ 3 \\ 85 \\ +\ \ 4 \\ \hline \end{array}$$

$42 + 18 = 60$
$17 + 3 = 20$
$85 + 4 = 89$
$60 + 20 + 89 =$ **169**

Or, keep a running total:
$42 + 18 = 60$
$60 + 10$ (from the 17) $= 70$
$70 + 7$ (from the 17) $+ 3 = 80$
$80 + 5$ (from the 85) $+ 4 = 89$
$89 + 80 =$ **169**

EXAMPLE 2

Find the volume.



$45 \times 11 \times 20$
$= 45 \times 20 \times 11$
$= 900 \times 11$
$=$ **9900 mm²**

EXAMPLE 3

Find 36% of 25

$= 25 \times (36 \times 0.01)$
$= (25 \times 0.01) \times 36$
$=$ 25% of 36
$=$ **9**

Mental Math

Grade 8 Mathematics

S–20

Sample Strategy

Properties and laws: Distributive property

$a \times (b + c) = a \times b + a \times c$
$a \times (b - c) = a \times b - a \times c$

EXAMPLE 1

7×92.1

7×92.1
$= (7 \times 90) + (7 \times 2) + (7 \times 0.1)$
$= 630 + 14 + 0.7$
$= \mathbf{644.7}$

EXAMPLE 2

Simplify:
$4(3 - 2x) + 1$

$= (4 \times 3) - (4 \times 2x) + 1$
$= 12 - 8x + 1$
$= \mathbf{13 - 8x}$

EXAMPLE 3

$9465 \div 11$

$9465 \div 11 = (8800 + 660 + 5) \div 11$
$8800 = 11 \times \mathbf{800}$
$660 = 11 \times \mathbf{60}$
$\frac{5}{11}$ **left over**

$= \mathbf{860\frac{5}{11}}$

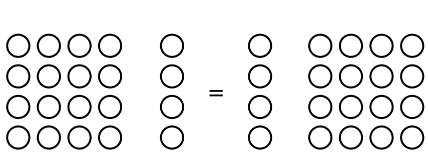
Teaching Strategies for Sample Strategies S–18 to S–20

Properties and laws: Commutative / Associative / Distributive

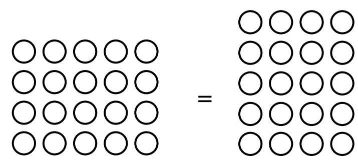
Use counters to develop specific rules about commutative, associative, and distributive properties.

Commutative Property

Show students a model of the commutative property with counters or blocks.



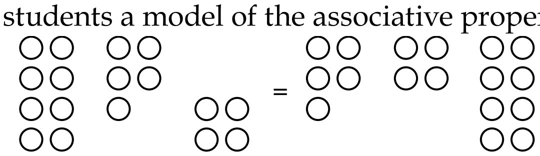
16 + 4 = 4 + 16



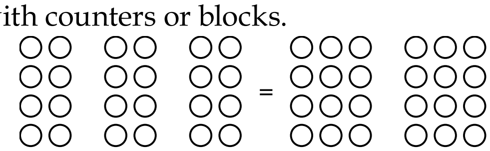
$4 \times 5 = 5 \times 4$

Associative Property

Show students a model of the associative property with counters or blocks.



8 + 5 + 4 = 5 + 4 + 8



$(2 \times 4) \times 3 = 2 \times (4 \times 3)$

- Have students try several examples of their own to determine if these properties will always hold true.
- Have students determine if these processes will work for subtraction and division, and explain why not.
- Have students develop rules with variables, such as:

  $a + b = b + a$

  $(a \times b) \times c = a \times (b \times c)$

- Have students use a number line to determine if the properties hold true when adding or multiplying whole numbers by negative integers.

Distributive Property

Have students analyze and explain why the following methods of multiplication work.

$$\begin{array}{r} 16 \\ \times\ 32 \\ \hline 12 \\ 20 \\ 180 \\ 300 \\ \hline \mathbf{512} \end{array}$$

$5 \times 4 = (4 \times 4) + (1 \times 4)$

$5 \times 4 = (3 \times 4) + (2 \times 4)$

$5 \times 4 = (2 \times 4) + (3 \times 4)$

$5 \times 4 = (1 \times 4) + (4 \times 4)$

$5 \times 4 = (2 \times 4) + (1 \times 4) + (2 \times 4)$

$5 \times 4 = (6 \times 4) - (1 \times 4)$

$5 \times 4 = (8 \times 4) - (3 \times 4)$

Mental Math

Grade 8 Mathematics

S–21

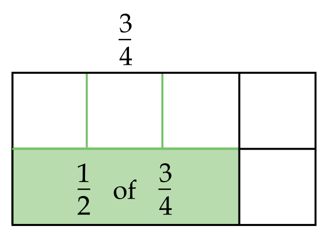
Sample Strategy

Advanced: Spatial reasoning/visualization

Forming a mental picture can be an effective way of reasoning through many math questions.

EXAMPLE 1

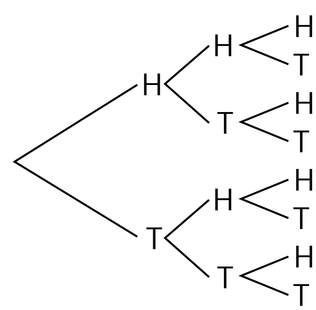
$$\frac{1}{2} \times \frac{3}{4}$$



$= \frac{3}{8}$

EXAMPLE 2

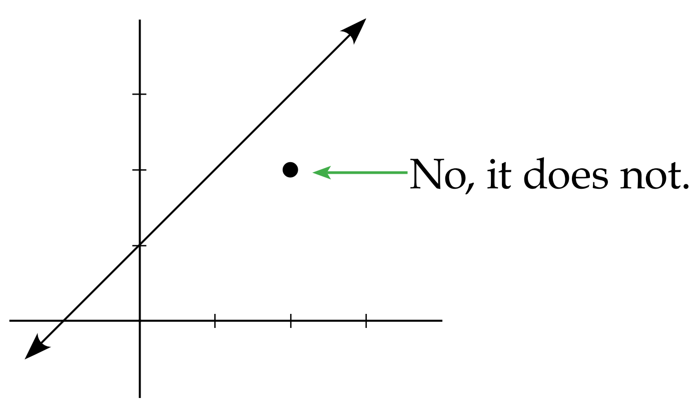
What is the probability of three tails in three coin tosses?



$= \frac{1}{8}$

EXAMPLE 3

Does the equation of the line $y = x + 1$ pass through (2, 2)?



Mental Math

Grade 8 Mathematics

S–22

Sample Strategy

Advanced: Hybrid approaches/A combination of strategies

With increased flexibility comes the ability to integrate multiple mental math strategies in the solution of a single problem.

EXAMPLE 1

52×16

$100 \times 16 = 1600$

$1600 \div 2 = 800$

Add 2 more groups of 16 ($16 \times 2 = 32$)

$800 + 32 = \mathbf{832}$

EXAMPLE 2

Calculate the cost of a \$14.99 dinner after a 15% tip.

Round \$14.99 to \$15, because 15% of 1¢ will be insignificantly small.

$$\$15 \times 15\% = (0.15 \times 10) + \frac{1}{2}(0.15 \times 10)$$
$$= 1.50 + 0.75$$
$$= 1.50 + 0.50 + 0.25$$
$$= \$2.25$$

$$\$2.25 + \$14.99 = \$2.25 + \$15.00 - \$0.01$$
$$\mathbf{= \$17.24}$$

EXAMPLE 3

Siju runs 5 metres per second. How fast is that in km per hour?

5 metres $\times$ 60 seconds = 300 metres per minute

$(30\cancel{0} \times 6\cancel{0}$ minutes$) \div 1\cancel{000}$ metres $= 3 \times 6$

= 18 km/hr.

Mental Math

Grade 8 Mathematics

S–23

Sample Strategy

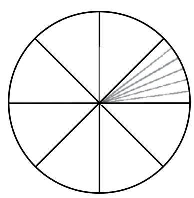
Memorization/automaticity

Once a conceptual understanding of basic facts, procedures, or formulas is developed, automaticity through memorization can be effectively used in order to make complex problem solving more efficient by freeing up working memory.

- Not all facts, formulas, or procedures need to be memorized, but for some it is important (i.e., $\pi \approx 3.14$, $50\% = \frac{1}{2}$, $8 \times 7 = 56$).
- Without a conceptual understanding, it becomes difficult for students to make connections across Middle Years topics.

EXAMPLE 1

$$\frac{1}{8} \times \frac{1}{6}$$



$= \frac{1}{48}$

Find one-sixth of one-eighth:

$8 \times 6 = 48$

OR

$8 \times 6 = (6 \times 6) + 12 = 48$

OR

$8 + 8 + 8 + 8 + 8 + 8 = 48$

Memorized process:

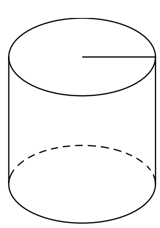
Multiply across

$1 \times 1 = 1$

$8 \times 6 = 48$ $\quad = \frac{1}{48}$

EXAMPLE 2

Explain two methods of finding the volume of the cylinder.



1. Memorized formula: $V = \pi r^2 h$
2. The volume of any right prism or cylinder has to be the area of the face times the height: $(3.14 \times r \times r) \times h = V$

Mental Math: Grade 8 Mathematics

Teaching Strategies for Sample Strategy S–23

S–23

Memorization/automaticity

Students prepare to commit many facts, procedures, and formulas to memory through regular and routine applications of efficient math strategies.

What about timed testing and flash cards?

- "Drill should only be used when an efficient strategy is in place." (Van de Walle and Folk, 169)
- Games can also be used to reinforce strategies, ultimately leading to automaticity.
- Although students may be able to memorize exclusively through drill work, without a foundation in conceptual understanding they will potentially lack the ability to make connections to future related areas of math.
- Forced memorization activities can make math appear to be an abstract system of complicated rules and procedures that are all dependent on each other. Flexibility, reason, and proofs don't fit well with this interpretation of math or with timed testing.
- Timed tests have been linked to math anxiety.
- Practice makes permanence. Repeated mistakes on timed tests can reinforce incorrect answers. Correct answers can also reinforce inefficient strategies.

What about memorizing procedures such as long division and double-digit multiplication?

- Students should understand why these processes work in order to make effective use of them. The standard long division and double-digit multiplication processes work because of place-value partitioning. The processes can be demonstrated and recreated with manipulatives. By reinforcing how these processes work, students can understand why they work and can better commit the processes to memory.
- Standard processes of long division and double-digit multiplication always work, but these processes are not often the most efficient method to solve a mental math problem, and they have little flexibility to them. Using these processes may demonstrate an ability to follow a rote procedure, but this does not necessarily demonstrate an understanding of the underlying math.

What about memorizing formulas?

- Grade 8 curriculum learning outcomes dealing with formulas all state that the formulas are to be "developed and applied" by students. This emphasis on student development of formulas and rules requires that students use their knowledge and reasoning skills to create formulas and clearly understand why those formulas work. Students may develop formulas that resemble the standard ones used, and they may ultimately find memorizing them to be an effective strategy.

Mental Math

Grade 8 Mathematics

S–24

Sample Strategy

Estimation strategies: Rounding using compatible numbers

Use nearly equal, simple-to-use values in order to perform mental math estimations efficiently and effectively.

Round to compatible numbers that are easy to compute mentally.

EXAMPLE 1

Approximate:

$$\begin{array}{r} 27 \\ 45 \\ 63 \\ +\ 81 \\ \hline \end{array}$$

$$\overbrace{27 + 45}^{\textit{about } 100} + \underbrace{63 + 81}_{\textit{about } 100} \approx \mathbf{200}$$

EXAMPLE 2

Estimate: $19.1 \div 3$

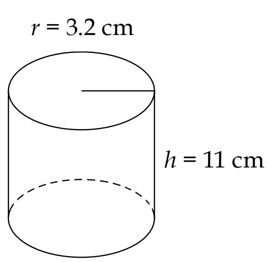
$19.1 \approx 18$

$18 \div 3 = \mathbf{6}$

Using the standard method of rounding would result in 20 ÷ 3, which is no easier to solve than 19.1 ÷ 3.

EXAMPLE 3

Will this can hold 250 mL of soup?



In this case, round down all values in order to ensure that the final estimation is an ***underestimate****.*

$V = \pi r^2 h$

$\approx 3 \times 9 \times 10 = \mathbf{270\ mL}$

Yes, this can will hold 250 mL of soup.

Mental Math: Grade 8 Mathematics

Background Information for Sample Strategies S–24 to S–27

Estimation strategies: Rounding and estimating

Methods of Rounding

There are many methods of rounding, and choosing the method to use depends entirely on the reason the rounding is being done. Students need to be familiar with different forms of rounding and should be able to reason through which method would be the best to use, depending on the context of the question.

Four Situations Where Rounding is Important

1. When using a simpler, nearly equal value to make mental estimation more efficient and communication simpler.

   Examples

   $\pi \approx 3.14 \left(\pi \approx 3, \text{ or } \pi \approx \frac{22}{7} \text{ may be even more useful in some instances}\right)$

   $\frac{2}{3} \approx 67\%$

   $227 \div 3 \approx 225 \div 3 = 75$

2. When communicating useful information.

   Examples

   After calculation, a price may appear to have three decimal places. The dollar system operates with only two decimal places. ($9.457 becomes $9.46.)

   100 students fit on each bus. How many buses are needed to transport 201 students? (201 ÷ 100 = 2.01 buses. Even though 2.01 buses is much closer to 2 than 3, 3 buses are required.)

3. When a calculation comes to a very precise result but displaying it in that way would present misinformation.

   Examples

   We round, and say that there are 7 billion people on Earth. (Reporting a precise number, such as 7 463 403 434, when that number is always changing, would present misinformation.)

   9 out of 10 dentists recommend a certain type of toothpaste. In a group of 15 dentists, how many recommend it? $\left(13\frac{1}{2}\right.$? Rounding it makes sense. There is no such thing as $\frac{1}{2}$ of a dentist.)

4. When exact numbers are not needed to answer a question.

   Example

   Will $67 be enough for four $13 apps? ($13 ≈ $15; $15 × 4 = $60, so $67 is enough.)

Mental Math

Grade 8 Mathematics

S–25

Sample Strategy

Estimation strategies: Common method—Half-round up

The most common method of rounding is to examine the value of the digit to the right of the one being rounded. If the digit is 1, 2, 3, or 4, we round down. If it is 5, 6, 7, 8, or 9, we round up.

Rounding results in a multiple of ten, which is often easy to work with.

EXAMPLE 1

Approximate: 94.2×67.8

$94.2 \times 67.8 \approx 90 \times 70$

$90 \times 70 =$ **6300**

EXAMPLE 2

Estimate: $2\,145\,648 - 389\,482$

$2\,145\,648 - 389\,482$

$\approx 2\,100\,000 - 400\,000 =$ **1 700 000**

EXAMPLE 3

Round to the nearest hundredth: 4012.235

$4012.23\underline{5}$

Because the underlined number is 5 or greater, we round the hundredths up to **4012.24.**

If 1, 2, 3, and 4 always round down and 5, 6, 7, 8, 9 always round up, could it ever become problematic that more numbers round up than down? If 5 is right in the middle, why does it round up all of the time? Also, what do you do with −7.5? Rounding up brings us closer to zero to −7! This rounding method is not perfect, but it can be very useful in certain situations and is widely used in industry and business.

Mental Math

Grade 8 Mathematics

S–26

Sample Strategy

Estimation strategies: Front-end rounding and estimation

There are several methods of front-end rounding and estimation.

- Keep the largest place value and truncate the rest.
- Round using the common half-round up method for the largest place value of each number.

EXAMPLE 1

Front-end estimation using the first method:

$36\,548 \times 712$

$\approx 30\,000 \times 700$
$= 30\,000 \times 700$
$= \mathbf{21\,000\,000}$

This method will always provide an underestimate.

EXAMPLE 2

Front-end estimation using the common half-round up method:

$36\,548 \times 712$

$\approx 40\,000 \times 700$
$= 40\,000 \times 700$
$= \mathbf{28\,000\,000}$

The second method of front-end estimation will always provide an equal or closer estimate to the first. In some cases this second method will provide an underestimate and, in others, an overestimate.

Mental Math

Grade 8 Mathematics

S–27

Sample Strategy

Estimation strategies: Round to the nearest 5

With the elimination of the penny from circulation in 2013, Canada has implemented a rounding system to the nearest 5¢ for cash payments.

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