Unit A: Problem Analysis

Half Course VI

Please see the print document for more activity suggestions. The document is available from the Manitoba Text Book Bureau (stock number 80354). To order, please visit <www.mtbb.mb.ca>.

HALF COURSE VI

Unit A: Problem Analysis

Hours: 7 in combination with Analysis of Games and Numbers

General Learning Outcome:

Develop and use mathematical strategies to solve problems in different situations.

The intent of this unit is to provide a range of interesting problems which call for a wide variety of strategies to solve them. These problems augment the work of other units and are to be embedded throughout the course.

Specific Learning Outcomes

A-1 Explain and solve problems using a variety of primarily non-algebraic approaches.

A-2 Describe the approach and the mathematics used in solutions to problems or activities.

PROBLEM ANALYSIS

Instructional Materials

- *Essentials of Mathematics 12*
- See Appendix I for possible activities.
- See Appendix II for additional resources.

PRESCRIBED LEARNING OUTCOMES

General Outcome

Develop and use mathematical strategies to solve problems in different situations.

Specific Outcome(s)

A-1 explain and solve problems using a variety of primarily non-algebraic approaches

A-2 describe the approach and the mathematics used in solutions to problems or activities

✔ **Communications**	Patterns
✔ **Connections**	✔ **Problem Solving**
Number Sense	✔ **Reasoning**
✔ **Organization and Structure**	Technology
	✔ **Visualization**

SUGGESTIONS FOR INSTRUCTION

Examples of non-algebraic approaches include geometry, networks, flow charts, organizational charts, simulations, etc.

Remember that for activities in this unit, the journey is more important than the destination. It is beneficial to discuss multiple approaches to solving these problems, particularly when the approaches have been developed by students. Are some approaches “better” than others? Why? On what grounds?

The problems contained in Appendix I are intended to provide material which is interesting in its own right and which complements the other units of the program. It is illustrative rather than exhaustive. Some activities have been chosen to illustrate a wide variety of job and consumer applications of mathematics that are largely non-algebraic. Others have been chosen because they are intrinsically interesting or because they challenge students to find and to use new ways of analyzing and thinking mathematically. All students do not need to engage in the same activities.

The activities in Appendix I are presented in **no** particular sequence. Teachers are encouraged to supplement this set of activities with material from other sources, such as the Internet. A preliminary list of possible resources is included in Appendix II.

It is suggested that these problems and activities be interspersed throughout the course as either extensions, enrichment, or a change of pace in the day-to-day work of the classroom. Some of them will link directly to particular units, but most are independent and **may** be used at any time. One approach would be to introduce problem analysis with a few days, possibly up to a week, of work on these activities. Intersperse the remainder throughout the course.

SUGGESTIONS FOR ASSESSMENT

Students' progress should be assessed over a length of time. Look, for example, for an increasing variety of problem-solving strategies and increasingly sophisticated explanations. Anecdotal records of how students work in pairs or groups on learning activities is appropriate. Well-developed solutions and reasoning could become part of a student's portfolio.

Complex problem-solving activities generally are not appropriate on pencil-and-paper timed tests.

SUGGESTED LEARNING RESOURCES

Print

Senior 4 Consumer Mathematics (45S) Part VI: A Course for Distance Learning. Winnipeg, MB: Manitoba Education, Training and Youth, 2002.
— Cover Assignments

Austin, J.D. *Applications of Secondary School Mathematics.* Reston, VA: NCTM, 1991.

Baron, C., et al. *Essentials of Mathematics 12.* Victoria, BC: British Columbia Ministry of Education, 2003. [ISBN 0-7726-4997-9]

Giblin, P., and I. Porteous. *Challenging Mathematics.* Toronto/New York: Oxford University Press, 1990.

Hirsch, C.R., and R.A. Laing. *Activities of Learning and Teaching*. Reston, VA: NCTM, 1993.

Mathematical Association of America and National Council of Teachers of Mathematics. *A Sourcebook of Applications of School Mathematics*. Reston, VA: NCTM, 1980.

National Council of Teachers of Mathematics. *NCTM Student Math Notes*. Reston, VA: NCTM, n.d.

Swetz, F., and J.S. Hartzler. *Mathematical Modeling in the Secondary School Curriculum.* Reston, VA: NCTM, 1991.

See Appendix II for a list of additional resources.

Appendix I

Teacher Information: Reading the News

Skills Required

- data collection
- sampling
- measurement
- planning a project

When to Use

This activity may be used at any time. Previous experience with collecting sample data will be helpful.

Teaching Information

This learning activity is probably best done in small groups. The various data collection activities can be divided up. Students will need to sample news reports from newspapers and TV newscasts. For the latter, recording a newscast and then transcribing is an appropriate strategy; more than one person is best for this. Have students plan beforehand how they will attack the problem. A full write-up should be expected.

Blackline Master: Reading the News

Many people get their news from the daily television news programs. Others read a newspaper such as the *Winnipeg Free Press* or the *Brandon Sun*. How many column inches of your favourite newspaper would be needed to print the complete text (not counting commercials) of a typical half-hour TV news program? [You will need to collect data for this. Look for good ways of gathering the information you need by careful sampling instead of counting **all** the words.]

Teacher Information: Covering a Roof

Skills Required

- spatial visualization
- calculating area
- use of calculator helpful

When to Use

Any time.

Teaching Suggestions

This problem requires visualization of the various parts of the roof, including the interior, in order to determine its height using the pitch of one-third. Encourage students to draw diagrams of the various vertical and slanted surfaces.

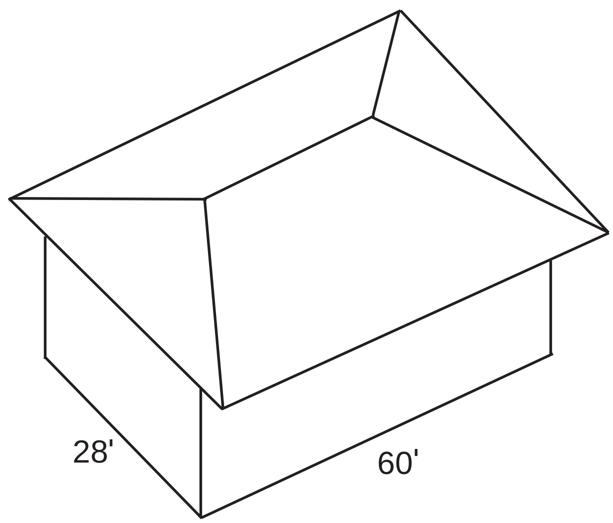
Solutions

The roof is 32' wide and 64' long at the eaves. Using the 16' half width and a pitch of one-third, the vertical height must be 5.33', and the slant height 16.87' for all surfaces. Thus, the ridge is 32'. Each trapezoidal area is 809.76 ≈ 810 square feet, and each end is approximately 270 square feet. Thus, the entire area is 2160 square feet. Each 4' × 8' sheet is 32 sq. ft.; the carpenter needs a minimum of 68 sheets.

Blackline Master: Covering a Roof

The cottage roof in the sketch below is to be covered with roof sheathing. The roof surfaces rise one foot for each three feet horizontally in a direction perpendicular to a wall; i.e., the pitch is one-third. The roof projects two feet horizontally beyond the walls (overhang is two feet).

How many square feet are needed? How many 4' x 8' sheets of sheathing should the builder purchase for the roof?



Teacher Information: Air Traffic Control with Transponders

Skills Required

- counting principle
- analysis

When to Use

Any time.

Teaching Suggestions

Sketches may assist students in visualizing the situation, both for the dials and for the directional considerations.

Solutions

1. $8^4 = 4096$ settings
2. Eight directional settings on dial 1; two settings on dial 2, and eight on each of the remaining: $8 \cdot 2 \cdot 8 \cdot 8 = 1024$

Blackline Master: Air Traffic Control with Transponders

Airplanes are equipped with electronic instruments called “transponders” which allow air traffic controllers to tell which airplane makes a given blip on their radar screens. The transponders have four dials calibrated from 0 to 7, and the controller asks a pilot to set his or her transponder at a given four-digit number. This number will then show up (in coded form) next to the blip made by the airplane.

1. How many different transponder settings are there? Is it possible that any one air traffic controller at any given time will have to give the same setting to two different airplanes? Why or why not?

2. Suppose air traffic controller Susan James decides to use the first digit of the setting to indicate the direction of a plane: 0 meaning north, 1 meaning northeast, 2 meaning east, etc., and the second digit to indicate whether an aircraft is approaching or departing. To how many aircraft can she assign distinct settings?

Teacher Information: Your Speed and the Car's Tires

Skills Required

- geometric visualization
- calculation with fractions and decimals
- measurement of circles

When to Use

Any time.

Teaching Suggestions

It may help students to think about the distance covered per unit time for wheels of different sizes which rotate at the same rpm (revolutions per minute). A speedometer calibration is based on the latter.

Solutions

1. Since your tire now has a smaller radius and a smaller circumference, it will travel less far with each revolution; in fact 1.374" less per revolution. Thus, you will be travelling at a slower speed than your speedometer indicates.

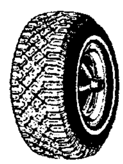
2. The increased circumference will move the car farther (1.963") per revolution; hence, your actual speed is faster than indicated by the speedometer.

3. If the tires are underinflated, their diameters and, hence, circumferences will be less, thus overstating your speed. Note that under- and overinflation of tires also affect car handling and tire wear.

Blackline Master: Your Speed and the Car's Tires

The speedometer on an automobile measures the rate at which the drive-shaft of the car is rotating, and, through the differential or transmission, the rate at which the wheels are turning. If the speedometer is accurate when the tires are new (9/32" tread depth) and properly inflated:

1. What will be the effect on your speedometer reading when the tires are worn to 1/16" tread depth (the legal minimum in many jurisdictions)?
2. What kind of change in the speedometer reading results from a change from worn tires (1/16" tread, properly inflated) to new snow tires (12/32" tread, properly inflated)?
3. Explain the speedometer error that might result from tires which are seriously underinflated.



Teacher Information: Rectangles and Diagonals

Skills Required

- geometric visualization
- factors, primes, and relatively prime dimensions
- pattern recognition

When to Use

Any time.

Teaching Suggestions

There are a variety of approaches to these questions, all of which may be equally valid. The learning activity can be done individually, in pairs, or in small groups. If students are not in groups, entering their data into a common database may make drawing conclusions easier. How students approach the problem is their responsibility. To direct students negates the intent of the exercise.

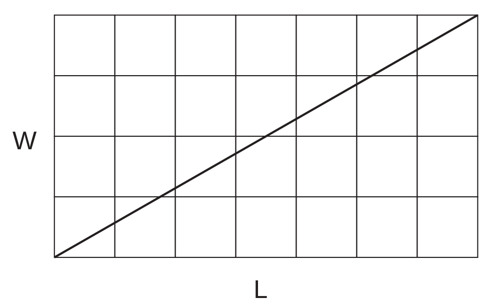
Solutions

Students will need to recognize that the dimensions should be considered in different ways. The dimensions may be:

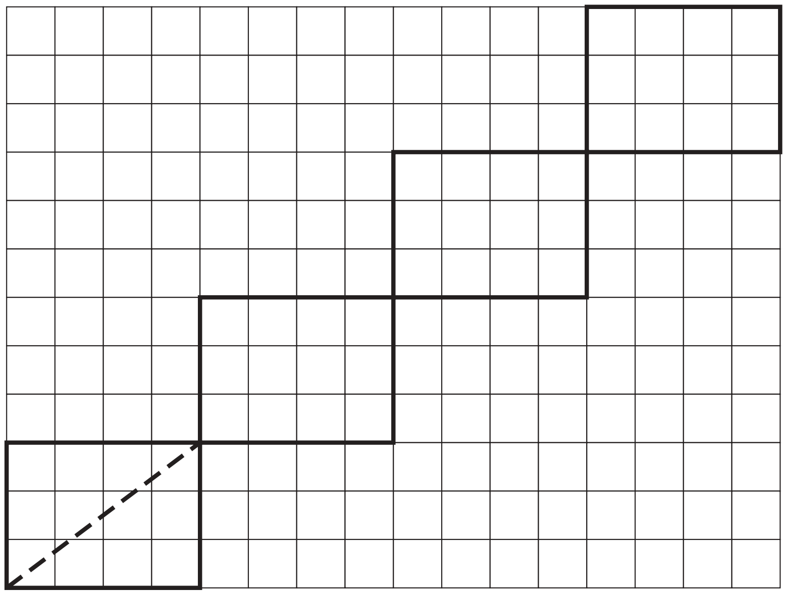
- relatively prime
- have a common factor
- have one dimension a factor of the other

If data are grouped as above, the relationships may be easier to see. Note that the following "solution" is phrased in algebraic terminology and notation. It is not the only way to approach the problem.

Relatively prime. In a 4 × 7 rectangle (the first example) the dimensions are relatively prime. The diagonal must cross seven squares going across. In some columns two squares will be crossed. Two squares in a column are crossed each time a horizontal line is crossed; i.e., three. Total squares crossed (T) is 7 + 3 = 10, or, to generalize: $T = L + W - 1$.

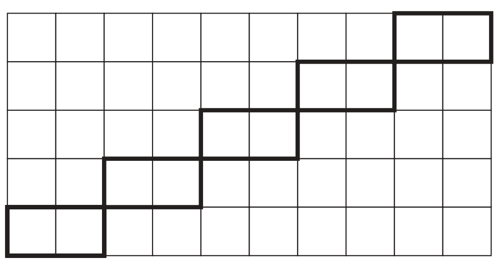


Common factor. Consider as an example the 12 × 16 rectangle. In this case the diagonals pass through points on the grid; hence, not crossing extra squares. In effect, there exist four smaller rectangles—4 being the greatest common factor. Consider one of the small rectangles:



Its dimensions are 3 × 4. These numbers are relatively prime so that the number of squares being crossed is: 3 + 4 – 1 = 6. This pattern repeats itself four times, and hence, the total number of squares crossed is 6 × 4 = 24. Algebraically, if the dimensions are *ax* and *ay* (a is the common factor), the number of rectangles is a, and the total number of squares crossed is: $a(x + y - 1)$ or $ax + ay - a$.

One dimension is a factor of the other. Consider the 5 × 10 rectangle.



This is a special case of the preceding example. The GCF is the number of small rectangles, and we therefore have 5(1 + 2 – 1) = 10 crossings. In this case the formula becomes the three cases together. The number of crossings is:

- L + W – 1
- $ax + ay - a$ but ax = L and ay = W
- $a + ay - a$ but a = L and ay = W

In the latter two cases, a is the greatest common factor. Thus, the above can be written:

- L + W – 1
- L + W – *a*
- L + W – *a*

The greatest common factor of two relatively prime numbers is 1. Thus, all of the above reduce to one relationship. The number of crossings is:

the sum of the dimensions less the greatest common factor of the dimensions

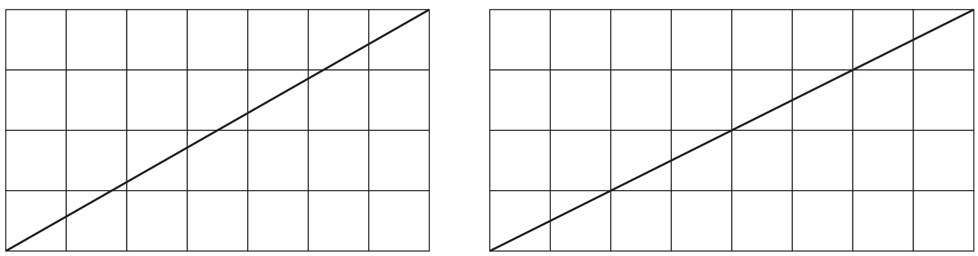
Blackline Master: Rectangles and Diagonals

Given a rectangle drawn on graph paper and a diagonal of that rectangle:

1. How many squares does the diagonal cross?
2. Can a relationship (descriptive or mathematical) be established between the dimensions of the rectangle and the number of squares crossed?

Write your findings, describing the relationship(s). Include diagrams such as those that are shown on this page.

It is up to you to decide how to tackle this problem. A few examples are drawn below to suggest that the dimensions of the rectangles should be considered in different ways.



What do you notice about the differences in the two examples shown above?

Teacher Information: Play-Offs

Skills Required

- experience with counting problems
- analysis
- tree diagrams (optional)

When to Use

Any time. May be most motivational when a play-off such as the World Series or the Stanley Cup is underway.

Teaching Suggestions

Remind students that a five-game play-off series ends as soon as one team wins three games; four games for a seven-game series. Some students may need to list all possibilities, and others may be able to visualize arrangements. For students more interested in hockey, the problem can be rephrased for the Stanley Cup. For the latter or for the World Series, an extension would be to include the play-off series leading up to the final series.

Solutions

1. If W stands for win and L for loss, here is a list of the possibilities. The series can be won in 10 possible ways.

W W W	1 way in 3 games
L W W W W L W W W W L W	3 ways in 4 games
L W W L W L W L W W L L W W W W L W L W W L L W W W W L L W	6 ways in 5 games

2. The World Series (best of seven games) can be won in 35 different ways.

Play-Offs: From *Problem of the Week* by Fisher, Lyle and Medigovich, William. © 1981 by Pearson Education, Inc., publishing as Dale Seymour Publications, an imprint of Pearson Learning Group. Used by permission.

Blackline Master: Play-Offs

It is common in sports that to win a play-off championship, a team must win the best out of five games. Whichever team wins three games first wins the series. In how many ways can a team win the play-off?

Problem 1

Suppose that you are the sports writer for your local or school newspaper. Determine the answer to the above problem and then write a short article explaining this to your loyal sports-fan readership.

Problem 2

In the World Series of baseball, a team must win four out of seven games to win the Series. In how many ways can this be done?

Did you know that . . .

- the World Series is named for the fact that the original series was sponsored by the *New York World* newspaper?

Teacher Information: Miscellaneous Problems

Skills Required

- visualization
- spatial reasoning
- pattern recognition

When to Use

Any time.

Teaching Information

- These problems could be handed out to students in question pairs or as an entire set.
- If handed out in pairs, students could hand in the solution to one out of the two problems.
- If handed out as an entire set, students could be asked to hand in solutions to two of the four questions. This would allow students an opportunity to choose problems that interest them or that they feel are easier.
- For Problem 1, students could work in pairs at the board. One student draws the shape while the other places a string at each vertex. The first student can draw the resultant diagonals on the board as they are appearing along the length of the string. Other examples of polygons with different numbers of sides could be used.
- For Problem 2, students could draw each layer and count the cannonballs. Once the total is reached, students could look for patterns.

Solutions

1. One can draw the actual polygons and count the diagonals, and determine a number pattern. It is then assumed in determining the number of diagonals for further polygons that this pattern continues. Another approach is to examine how the diagonals are formed and develop a more generalized approach. Consider the hexagon first. From each vertex of the hexagon, we can draw diagonals to three vertices. Since there are six vertices, there are $\frac{(6)(3)}{2} = 9$ diagonals. (We divide by 2, since each diagonal would otherwise be counted twice.) For the heptagon, we can draw diagonals to four vertices. Thus, there are $\frac{(7)(4)}{2} = 14$ diagonals. This procedure can be generalized. If a convex polygon has n vertices, from each vertex $(n - 3)$ diagonals can be drawn, one to each vertex except the vertex itself and its immediate neighbours. Thus an n-sided polygon has $\frac{(n)(n-3)}{2}$ vertices.

2. Consider the diagram below which shows the bottom layer of cannonballs.

   The bottom layer contains 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

   The next layer contains 1 + 2 + 3 + 4 + 5 + 6 = 21

   And so on: 1 + 2 + 3 + 4 + 5 = 15

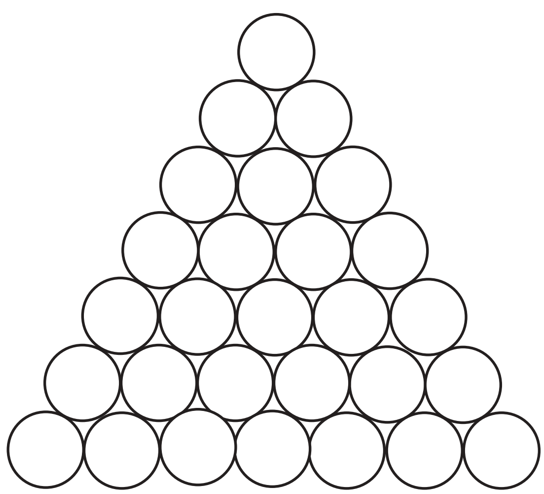
   1 + 2 + 3 + 4 = 10

   1 + 2 + 3 = 6

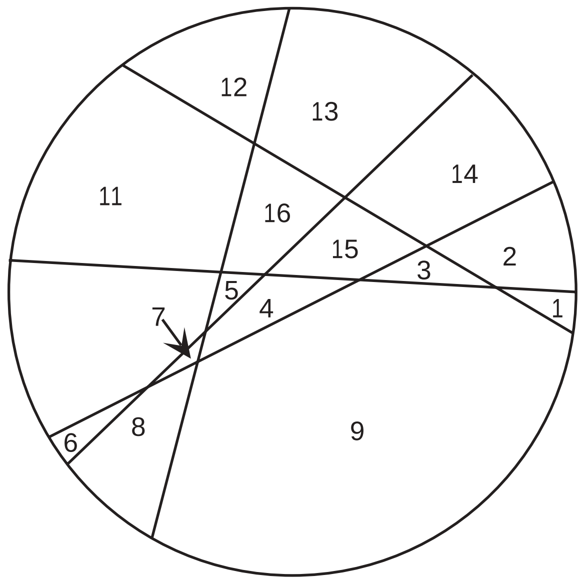
   1 + 2 = 3

   The top layer contains one cannonball.

   Altogether, the pyramid contains 28 + 21 + 15 + 10 + 6 + 3 + 1 = 84 cannonballs.



3. To maximize the number of pieces, every cut must intersect every other cut, and no more than two cuts can pass through any given point. Then there will be 16 pieces as shown.



4. A 36" pizza has a diameter of 36" and a radius of 18". Therefore, its area is:

$$A = \pi r^2 = 324\pi \text{ sq. inches.}$$

For an 18" pizza, $r = 9$. Therefore, $A = 81\pi$ square inches for one pizza and 162π square inches for two pizzas, only half as much as for one 36" pizza—not a very good deal!

Remember: Twice the radius results in four times the area.

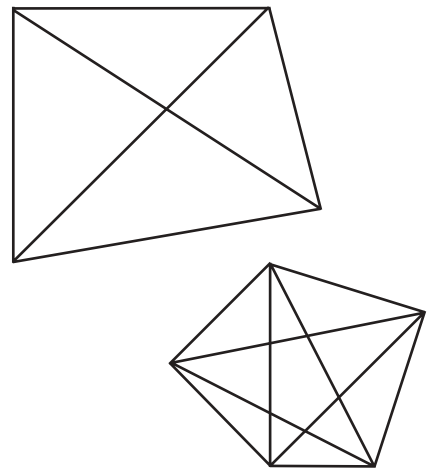
Blackline Master: Miscellaneous Problems

1. A quadrilateral has two diagonals, while a pentagon has five diagonals. Find the number of diagonals in:

   a) a hexagon (6 sides)

   b) a heptagon (7 sides)

   c) a dodecagon (12 sides)



2. At an historical fort, cannonballs are arranged in the form of an equilateral triangle with seven cannonballs to a side. Another layer, in the form of an equilateral triangle with six cannonballs to a side, is placed on top of it. This process is continued to form a pyramid with one cannonball on top. How many cannonballs are in the pyramid?

3. What is the largest number of pieces into which you can cut a circular pizza with five straight cuts? (You cannot stack the pieces.) Explain your answer.

4. Pizza Plaza advertises that it sells the largest circular pizza in the city, 36 inches in diameter. At a special price of $35.99, they claim it is your best deal. You and your friends are really hungry and have decided to order one of these. When the delivery man arrives, he tells you that they are having trouble with their large oven and could not bake the 36" pizza. Instead, he has brought you two 18" pizzas for the same price. Is this a fair exchange? Why or why not?

Miscellaneous Problems: Manitoba Education and Training. *Problems for High School Mathematics: Support Document*.

Appendix II

Additional Resources

Print

The Association of Teachers of Mathematics. *Eight Days a Week: Puzzles, Problems and Questions to Activate the Mind*. The Association of Teachers of Mathematics. ISBN 1-898-611-09-2.

Brecker, Erwin. *Lateral Logic Puzzles*. Sterling Publishing Company, Inc. ISBN 0-8069-0618-9.

Bremner, John. *Mensa Maths Wizards for Kids*. Carleton Books Limited. ISBN 1-85868-555-9.

Carter, Philip, Ken Russell, and John Bremner. *The Ultimate Puzzle Challenge*. Carlton Books Limited. ISBN 1-85868-716-0.

Crisler, N., P. Fisher, and G. Froelich. *Discrete Mathematics through Applications*. New York, NY: W.H. Freeman, 1994.

DeSpezio, Michael A. *Giant Book of Challenging Thinking Puzzles*. Sterling Publishing Company, Inc. ISBN 0-8069-2087-4.

Dossey, J.A., M. Kenney, et al. *Discrete Mathematics*. Glenview, IL: Scott Foresman, 1987.

Forte, Imogene, and Sandra Schur. *180 Icebreakers to Strengthen Critical Thinking and Problem-Solving Skills*. Incentive Publications, Inc. ISBN 0-86530-345-2.

Graham, Evelyne M. *Think-A-Grams*. Critical Thinking Press and Software.
ISBN Numbers: Book A1: 0-89455-329-1
Book A2: 0-89455-430-1
Book B1: 0-89455-330-5
Book B2: 0-89455-431-X
Book C1: 0-89455-331-3
Book C2: 0-89455-432-8

Hunter, J.A.H. *Entertaining Mathematical Teasers and How to Solve Them*. Dover Publications, Inc. ISBN 0-486-24500-4.

Kenney, M.A. *Lesson in Mathematical Doodling*. Boston, MA: Boston College Press, 1976.

Mathematics in School. The Mathematical Association, 259 London Road, Leicester, UK: LE2 3BE.

The Mathematics Teacher. National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA: 22091-1593.

Matt-Smith, Geoffrey. *Mathematical Puzzles for Beginners and Enthusiasts*. Dover Publications, Inc. ISBN 0-486-20198-8.

Maurer, S.B., and A. Ralston. *Discrete Algorithmic Mathematics*. Reading, MA: Addison-Wesley, 1991.

Nash, Helen, and Dorothy Masterson. *Humorous Cryptograms*. Sterling Publishing Company, Inc. ISBN 0-8069-3982-6.

National Council of Teachers of Mathematics. *How to Evaluate Progress in Problem Solving.* National Council of Teachers of Mathematics. ISBN 0-87353-241-4.

Sloane, Paul, and Des MacHale. *Improve Your Lateral Thinking*. Sterling Publishing Company, Inc. ISBN 0-8069-1374-6.

Weber, Ken. *Five Minute Mysteries for the Armchair Detective*. Stoddart Publishing Co., Ltd. ISBN 0-7737-5210-2.

Williams, J. "Graph Coloring Used to Model Traffic Lights." *Mathematics Teacher* 85 (March 1992): 212-14.

Internet

There are many sites on the Internet with problems and puzzles. If you are using a search engine to find these sites, search using the words "Mathematics Puzzles Problems."

As of February 2004, the following sites were available:

AAA Math

<http://www.aaamath.com>

This site has games and practice sheets for various grade levels and topics. There are links to other sites on the web with games and puzzles.

Math Forum

<http://mathforum.org>

This is a good site to begin searching for problems and puzzles. One feature is **Problems of the Week**. New problems are available as well as a library of previous problems. Students can submit their answers and get some feedback. There are links to other math sites and several departments that are useful.

Word Problems for Kids

<http://www.stfx.ca/special/mathproblems/welcome.html>

This is a Canadian site with word problems, hints, and solutions from previous mathematics competitions. The problems are sorted by grade level. Choosing problems from Grades 5 through 9 will lead to a wealth of non-algebraic problems.